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Earth-Flattening Approximation for Body Waves Derived from Geometric Ray Theory – Improvements, Corrections and Range of Applicability *

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Abstract. A new derivation of the earth-flattening approximation (EFA) for body waves from geometric ray theory is given which results in an improved version of the EFA. This version agrees with the EFA, derived by Chapman (1973) from wave theory. Moreover, it allows absolute, not only relative, body-wave amplitude calculations for given source time functions. The choice of the density transformation of the EFA is shown, by numerical calculations, to be uncritical for body-wave amplitudes in the period range up to 30 s. An error in an earlier derivation of the EFA (Müller, 1973a) is corrected. This error requires a new investigation of the range of applicability of the EFA, which is performed for the P-wave propagation through a homogeneous sphere. The results are similar to those of the earlier paper: long-period *P* waves with dominant periods up to about 20 s can be treated practically exactly, as long as they do not pass closer than about 800 km to the earth's center.

Key words: Earth-flattening approximations – Geometric ray theory – Wave theory

Introduction

Earth-flattening approximations (EFAs) for body waves have been derived from geometric ray theory (Müller, 1971, 1973a) and from wave theory (Chapman, 1973; see also Gilbert and Helmberger, 1972; Helmberger and Harkrider, 1972; Hill, 1972). Although the basic structure of both EFAs is the same, there is a notable difference: according to the EFA of Müller (1973a) (in the following called paper *I*) the velocity-density-depth distribution in the flat earth depends both on source and receiver radius whereas Chapman's EFA is independent of these radii. Because of the dependence on source and receiver radius the EFA of *I* requires frequency and time transformations, as long as these radii are different, which is not required with Chapman's EFA. The essential assumption in the derivation of the EFA of *I*, which entails its more complicated form, is that medium properties at the source should agree in the spherical and the flat

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earth, such that the radiation of P and S waves is the same. This condition can be relaxed, and it is shown in this paper that the resulting EFA, derived from geometric ray theory, agrees with Chapman's EFA in its essential features. Moreover, it allows the calculation of absolute (not only relative) body-wave amplitudes for a given source time function, e.g., the moment function in the case of a double-couple point source.

The density transformation of the EFA is not well determined in the case of P-SV waves, neither by geometric ray theory (which is not surprising) nor by wave theory. Therefore, numerical calculations of theoretical seismograms for different density transformations are desirable, and they should show whether or not the choice of the density transformation is critical for practical purposes. Results of calculations for the mantle P phase and the core reflection PcP are discussed in this paper.

Another purpose of this paper is to correct an error in *I*, related to the amplitude correction factor for diffracted waves. A correct application of geometric ray theory gives the same $(\Delta/\sin\Delta)^{1/2}$ factor as for other body waves (Δ = epicentral distance).

Finally, a new investigation of the range of applicability of the EFA is performed by calculating the P-wave propagation from an explosive point source through a homogeneous sphere. Exact results are available in this case, since it corresponds to propagation through a homogeneous unbounded medium. This test has already been used by Helmberger (1973) for P waves propagating as deep as 1150 km in a sphere of the size of the earth. These calculations are extended here to much greater depths in order to find out for which wavenumber times radius products the EFA still works with sufficient accuracy.

Theory

The following is a summary of the properties of rays in a sphere and a half-space, according to geometric ray theory. Most notations are explained in Figure 1, and subscripts *s* and *f* refer to the spherical and the flat earth, respectively. The formulas are given for the simple type of ray shown in Figure 1, but the results derived from them are also true for other types such as rays with a turning point or reflected rays.

Spherical earth:

Epicentral distance: $$x_s = P_s \int_{r_1}^{r_0} \frac{r_1}{r} \left(\frac{r^2}{V_s^2} - P_s^2\right)^{-1/2} \mathrm{d}r \tag{1}$$

Travel time: $$t_s = \int_{r_1}^{r_0} \frac{r}{V_s^2} \left(\frac{r^2}{V_s^2} - P_s^2\right)^{-1/2} \mathrm{d}r \tag{2}$$

Ray parameter: $$P_s = \frac{r_0 \sin\psi_0}{V_s(r_0)}$$

Amplitude: $$A_s = \left(\frac{\rho_s(r_0)\, V_s(r_0)\, a^2 \sin\psi_0}{\rho_s(r_1)\, V_s(r_1)\, r_1 \sin\Delta \left|\frac{\partial x_s}{\partial \psi_0}\right|_r \cos\psi_s} \right)^{1/2} A_{s0} \tag{3}$$

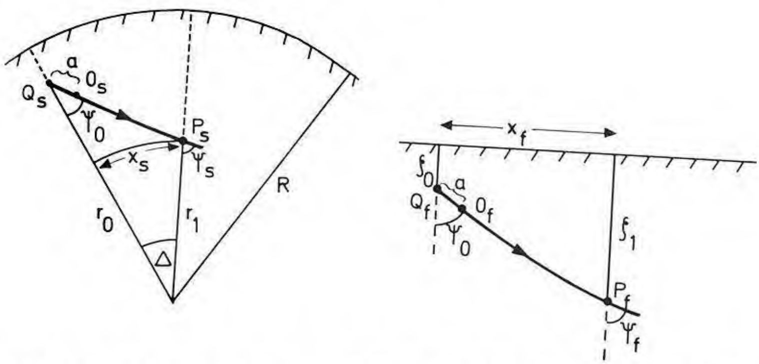

Fig. 1. The spherical earth (left), the flat earth (right) and two corresponding rays from the source $Q_{s,f}$ to the receiver $P_{s,f}$. The radiation angle is the same for both rays

$V_s(r)$ = P or S velocity
$\rho_s(r)$ = density } at radius r

A_{s0} = amplitude at a reference point O_s on the ray with distance a from the source Q_s

Flat earth:

Epicentral distance: $$x_f = P_f \int_{\zeta_0}^{\zeta_1} \left(\frac{1}{V_f^2} - P_f^2\right)^{-1/2} d\zeta \tag{4}$$

Travel time: $$t_f = \int_{\zeta_0}^{\zeta_1} \frac{1}{V_f^2} \left(\frac{1}{V_f^2} - P_f^2\right)^{-1/2} d\zeta \tag{5}$$

Ray parameter: $$P_f = \frac{\sin\psi_0}{V_f(\zeta_0)}$$

Amplitude: $$A_f = \left(\frac{\rho_f(\zeta_0)\, V_f(\zeta_0)\, a^2 \sin\psi_0}{\rho_f(\zeta_1)\, V_f(\zeta_1)\, x_f \left|\frac{\partial x_f}{\partial \psi_0}\right|_{\zeta_1} \cos\psi_f} \right)^{1/2} A_{f0} \tag{6}$$

$V_f(\zeta)$ = P or S velocity
$\rho_f(\zeta)$ = density } at depth ζ

A_{f0} = amplitude at a reference point O_f on the ray with distance a from the source Q_f.

The depth and velocity transformations of Gerver and Markushevich (1966),

$$\zeta = R \ln \frac{R}{r}, \qquad V_f(\zeta) = \frac{R}{r} V_s(r), \tag{7}$$

which are independent of source and receiver radius yield, when inserted into (1) and (2),

$$x_s = \frac{r_1}{R} x_f \quad \text{or} \quad x_f = R\Delta \tag{8}$$

and

$$t_s = t_f.$$

Here, (4) and (5) have been used. It follows that the mapping of the sphere onto the half-space is independent of r_1, and that the travel times between corresponding points in the sphere and the half-space agree for arbitrary values of r_0 and r_1; in both regards the EFA derived here differs from the EFA of *I*. The constancy of ray parameter along the ray in both media yields $\psi_s = \psi_f$. Inserting this and (7) and (8) into (3) one obtains, using (6):

$$A_s = \left(\frac{\Delta}{\sin\Delta}\right)^{1/2} \frac{R}{r_1} \left(\frac{r_0\,\rho_s(r_0)\,\rho_f(\zeta_1)}{r_1\,\rho_s(r_1)\,\rho_f(\zeta_0)}\right)^{1/2} \frac{A_{s0}}{A_{f0}} A_f.$$

A density transformation similar to the velocity transformation,

$$\rho_f(\zeta) = \left(\frac{R}{r}\right)^n \rho_s(r), \tag{9}$$

where according to wave theory n depends on the wave type investigated, yields finally the amplitudes in the sphere in terms of the amplitudes in the half-space:

$$A_s = \left(\frac{\Delta}{\sin\Delta}\right)^{1/2} \left(\frac{R}{r_1}\right)^{\frac{n+3}{2}} \left(\frac{r_0}{R}\right)^{\frac{n+1}{2}} \frac{A_{s0}}{A_{f0}} A_f. \tag{10}$$

The relation between the exponent of receiver radius r_1 and the exponent n in the density transformation (9) is the same as with Chapman's EFA. Thus, both EFAs agree in all essential points.

The amplitude correction formula (10) allows the calculation of absolute amplitudes in the sphere, provided the ratio A_{s0}/A_{f0} of the initial amplitudes is known. In the framework of geometric ray theory these amplitudes have to be considered as the amplitudes of the far-field term of displacement, taken close to the source. The simplest assumption is $A_{s0}/A_{f0} = 1$; this means that the source is described by its (far-field) displacement-time function. There may, however, be cases where one prefers to describe a double-couple source by its moment function $M(t)$ or an explosion by its excitation function (or reduced displacement potential) $F(t)$. From the far-field displacements of these sources in a homogeneous unbounded medium, one derives for a double-couple

$$\frac{A_{s0}}{A_{f0}} = \frac{\rho_f(\zeta_0)\,V_f^3(\zeta_0)}{\rho_s(r_0)\,V_s^3(r_0)} = \left(\frac{R}{r_0}\right)^{n+3}$$

and for an explosion

$$\frac{A_{s0}}{A_{f0}} = \frac{V_f(\zeta_0)}{V_s(r_0)} = \frac{R}{r_0}.$$

Discussion

As paper *I* and this paper show, geometric ray theory allows the construction of different EFAs. From a theoretical point of view preference should be given to the EFA derived here, since it agrees with an EFA from wave theory, which has not yet been shown for the EFA of *I*. From a practical point of view, no essential differences exist, as follows from calculations of theoretical seismograms. For example, theoretical long-period P and PcP phases between 40° and 70° have been computed by Müller et al. (1977, Fig. 7) for the Jeffreys-Bullen earth model and a source at a depth of 600 km, using the EFA of *I*. Recalculation with the revised EFA shows agreement generally within 1% in absolute amplitudes and in the amplitude ratio PcP/P.

The density transformation of the EFA is not well defined from geometric ray theory. Chapman (1973) has shown from wave theory that in (9) $n=1$ is optimum, although not exact, for P waves in liquid media, i.e., for the acoustic case, and $n=-5$ for SH waves. For P-SV waves in solid media no optimum value could be found. Numerical calculations of theoretical seismograms, again for long-period P and PcP between 40° and 70° from a deep source, show changes in amplitudes from $n=1$ to $n=-5$ which do not exceed 2% in the case of P and 4% in the case of PcP. These numbers decrease for a closer approximation of the velocity-density-depth distribution by layers (which in the computational method used, the reflectivity method (Fuchs and Müller, 1971), are homogeneous in the flat earth and hence inhomogeneous with negative velocity gradients in the spherical earth). The conclusion from this is that for practical purposes the choice of n is not critical in body-wave studies. On this background, a theoretical argument can be made in all three cases (acoustic, SH and P-SV) in favor of $n=-1$. It is an experience from numerical calculations that the influence of density on body-wave amplitudes is strongest for vertically travelling waves. For these the controlling parameter is the impedance, i.e., the product of velocity and density. Therefore, it is reasonable to match the impedances of the spherical and the flat earth, which means $n=-1$. For waves travelling predominantly horizontal this value is as reasonable as any other from -5 to 1.

The amplitude correction formula (10) applies also in the case of diffracted rays, contrary to what was stated in *I*; i.e., formulas (12) and (17) of *I* are wrong. The simplest argument is that a diffracted ray which runs parallel to the diffracting boundary can be approximated arbitrarily close by a ray of the type discussed so far, having a turning point. This is done by introducing in an arbitrarily thin layer above the diffracting boundary a velocity gradient $dV_s/dr = V_s(r_d)/r_d$ where r_d is the radius of the diffracting boundary and $V_s(r_d)$ the velocity directly above it. Then, (10) is applicable. For finite frequencies, i.e., non-zero wavelengths, the original and the new velocity distribution are equivalent, and thus (10) is also valid for diffracted rays. This qualitative argument is confirmed by strict geometric-ray-theory calculations of the changes in wave amplitude along the segments of a truly diffracted ray in the spherical earth and its image in the flat earth. The error in *I* is due to a wrong sequence in the treatment of the ray segments.

A consequence of the error is that in Müller (1973b) theoretical P_{diff} amplitudes are slightly incorrect (see Müller, 1976). As a more serious consequence, it seems no longer certain that the EFA can be applied in those body-wave propagation problems for which the product wavenumber times radius, kr, is greater than about 16. This lower limit had been inferred in *I* from a comparison of exact P_{diff} amplitudes with those following from calculations via the EFA, including the wrong amplitude correction factor for diffracted rays (*I*, Fig. 3). If the correct factor is used the agreement is less good, and hence one would derive a greater minimum value of kr and consequently a more restricted range of applicability of the EFA. This question requires further investigation which is reported in the next section.

Range of Applicability of the EFA

The range of applicability of the EFA can be tested by comparing theoretical seismograms for a model, for which they can be calculated analytically, with those calculated numerically via the EFA and the reflectivity method for a layered half-space. The simplest test model is a homogeneous sphere with an explosive point source and receivers at the surface. The radius of the sphere is assumed to be 6370 km, the P velocity 10.00 km/s, the *S* velocity 5.77 km/s and the density 5.50 g/cm^3. In the corresponding flat medium the wave velocities increase exponentially with depth; the density decreases exponentially with depth, according to the exponent $n=-1$ in the density transformation (9). For application of the reflectivity method, this half-space is approximated by homogeneous layers which corresponds to saw-tooth-like velocity and density-depth distributions in the sphere. The thickness of the inhomogeneous layers in the sphere is 50 km in a first calculation; it is reduced in a second calculation to 25 km at depths greater than 4000 km and to 12.5 km below 5000 km, in order to test whether or not the approximation of the homogeneous sphere is sufficient. Both calculations give essentially the same results, the differences in the maximum peak-to peak amplitudes being 2% or less.

The far-field term of the displacement of the spherical P-wave at a reference distance close to the source is assumed to be

$$s(t)=\sin 2\pi \frac{t}{T}-\tfrac{1}{2}\sin 4\pi \frac{t}{T}, \qquad 0\leqq t\leqq T=20\,s. \tag{11}$$

The dominant wavelength is about 180 km in the sphere. The epicentral distance of the receivers increases from 120°–170°, such that the rays from the source to the receivers pass closer and closer to the center of the sphere where the EFA definitely breaks down. The influence of the free surface is disregarded, both at the source and at the receivers; in reality the test is one for an unbounded medium for which theoretical seismograms can be calculated analytically in a well-known manner from (11).

Figure 2 shows a comparison of exact theoretical seismograms for the displacement component along the ray from the source to the receiver with calculations using the EFA. The component perpendicular to the ray, which in theory vanishes, has maximum amplitudes less than 1% of those of the component along the ray. The agreement between analytical and numerical calculations is very good up to 150°. Then discrepancies gradually develop, and at 170° both amplitudes and pulse forms are significantly different. The most

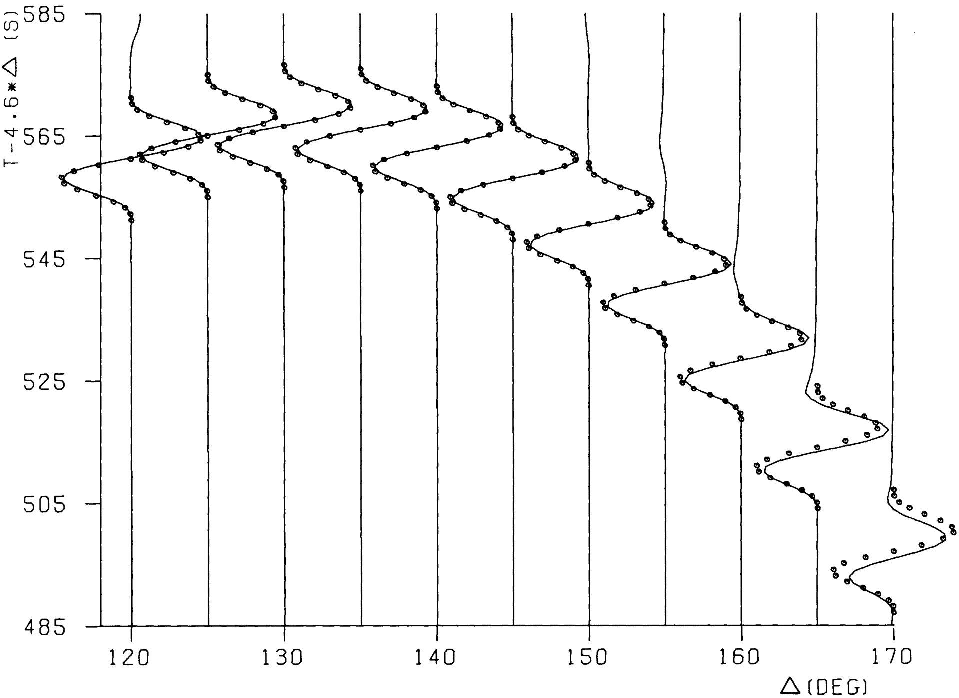

Fig. 2. P-wave propagation from an explosive point source through a homogeneous sphere: comparison of exact theoretical seismograms (circles) and numerical calculations, based on the earth-flattening approximation and the reflectivity method for a half-space (solid lines). For more details see text

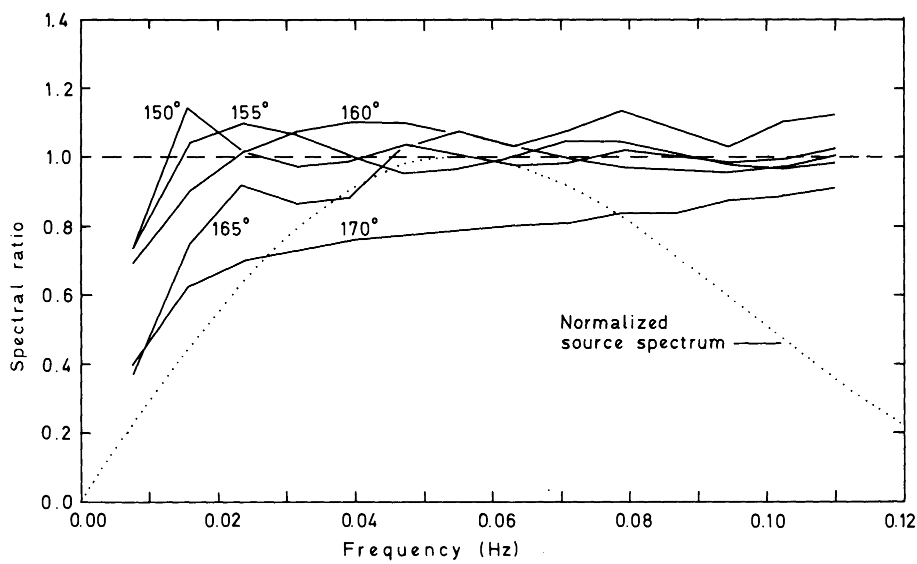

Fig. 3. Ratio of the amplitude spectra of the numerically calculated and the exact seismograms of Figure 2, for epicentral distances from 150°–170°. The length of the time interval considered is 127 s, and the time step 1 s. The normalized source spectrum follows from (11)

important conclusion can already be drawn by inspection from the seismograms at 165° where, inspite of certain differences in pulse form, the maximum peak-to-peak amplitudes are practically identical. The distance of the turing point of the ray from the center of the sphere is 831 km, which in the real earth corresponds to a depth of about 400 km below the boundary of the inner core. Since the velocity-depth distribution in the inner core is quite similar to the one in the homogeneous sphere under investigation, it is safe to conclude that the EFA can be applied without essential errors in amplitude studies of long-period core phases with dominant periods up to at least 20 s, provided that the waves do not propagate deeper than about 400 km below the inner-core boundary. Moreover, spectral analysis of the seismograms of Figure 2 shows (Fig. 3) that at low frequencies the EFA leads to systematically reduced spectral ratios of the numerically calculated to the analytical seismograms. At 165° the spectral ratio is in error by more than 10% at periods greater than about 30 s, taking a smoothed version of the spectral-ratio curve. Considering 10% as an acceptable error in the computational method, compared with the normally much larger observational error in amplitude studies of long-period waves, one derives $kr \geqq 17$ as the admissible range of the product kr in studies of long-period body-wave amplitudes with the EFA. To be on the safe side, the waves should not pass closer than about 800 km to the earth's center. In essence, these are the same conclusions that had been reached in *1*.

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