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Scattered Waves in the Coda of *P*

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Abstract. This paper presents a survey of the development and use of first order elastic scattering theory in seismology. The various methods used to provide expressions for scattered waves from variations in structure are shown to lead to a single scattering formula.

A ray theory approximation for the incident and scattered waves provides a simple formula from which the radiation patterns of different types of scatterer can be derived. As an illustration, the solution for a homogeneous 'average' structure is given in detail.

The statistical properties of the signal in time are clearly related to those of the scatterers in space and, in particular, the correlation time of the signal is related to the correlation distance of the scatterers.

The paper ends with a discussion of the possible use of first order (weak scattering) theory in cases when the scattered signals are large.

Key words: Elastic scattering theory – Scattering in the earth.

1. Introduction

The idea of scattering from slight inhomogeneities in the Earth's structure, as an explanation of particular phases of a seismogram, has become widespread in recent years.

In 1965 Wesley applied theoretical calculations of scattered waves from inhomogeneities in lithospheric structure to account for the coda of *P* in records of nearby explosions. This was followed by similar studies of small earthquakes by Aki (1969), Takano (1971) and Aki and Chouet (1975). Greenfield (1971) brought in the rough topography of the surface near the source to explain the coda of teleseismic *P* from Novaya Zemlya explosions. More recently, King et al. (1975) and Cleary et al. (1976) applied scattering theory to account for the coda of *P*, including precursors to *PP*, at epicentral distances of around 100°, in terms of surface reflections of *P* which have deviated from their usual ray paths as a result of crustal and upper mantle heterogeneity. A similar mechanism has been proposed, to account for precursors to *PKPPKP*, by King and Cleary (1974) and Haddon et al. (1976).

In 1972, Haddon suggested that scattering from inhomogeneities in the lower mantle might be the explanation of precursors to *PKIKP*. Comparisons of the data with theoretical calculations appeared in papers by Cleary and Haddon (1972), Doornbos and Vlaar (1973), King et al. (1973), Vinnik (1974), Doornbos (1974), King et al. (1974), Haddon and Cleary (1974), Wright (1975), Doornbos (1976) and Husebye et al. (1976).

In addition to the generation of wave trains by scattering, small irregularities in Earth structure will cause variations in amplitudes, arrival times and phase velocities of commonly observed seismic phases. While such fluctuations must of course be accounted for in any comprehensive theory, this paper is concerned only with the scattered waves which can be measured separately from the main phase, as in the codas and precursors mentioned above.

On the assumption that the incident rays are scattered once only before arriving at an observation point, simple kinematic theory has been used in many cases to predict the onset times of scattered waves, as well as the variation of phase velocity along the wave train. Multiple scattering needs to be taken into account if the scattering is strong, but the complexity of the calculations would, almost inevitably, be very much increased.

In order to estimate the amplitude of the scattered waves in terms of the properties of the solid medium, full elastodynamic theory must be used. The theory underlying almost all calculated expressions is a first order perturbation approximation; that is, the scattered wave field is assumed to be a small perturbation on the primary waves.

Expressions for first order scattering in an elastic medium were first obtained by Miles (1960), who treated the time-harmonic problem of weak scattering from a small heterogeneous region. At the same time Gilbert and Knopoff (1960) presented a method of dealing with small variations in surface topography. Expressions for the scattered field were also derived by Herrera and Mal (1965) for a volume distribution of weak scatterers, and by Herrera (1965) for thin scatterers.

When the size of individual scatterers within a heterogeneous region is small compared with the size of the region, the details of the structure are of little importance and one may expect only to infer the statistical properties of the elastic parameters. Theoretical results have been obtained (Knopoff and Hudson, 1964; Hudson and Knopoff, 1967) for the mean square amplitudes of scattered waves where the incident wave is time harmonic and the mean is calculated for a statistical ensemble of scattering regions. In a similar calculation, Dunkin (1969) derived expressions for the correlation function of the signals received by 2 observers from an incident spherical *P* wave.

Some of the theoretical results used to compare with seismic observations have been derived from acoustical theory. Vinnik (1974), for instance, made his calculations on this basis. However, the acoustical model is unsatisfactory for several reasons. Most of the calculations made by Haddon and his co-workers are based on the elastodynamic equations and first-order perturbation theory, rederived by Haddon but unpublished. The published results (see, for instance, Haddon and Cleary, 1974) correspond to mean square amplitudes of waves scattered from an incident wave whose time variation is either simple harmonic or a random time series, modified by a slowly varying envelope. Doornbos (1976) too has provided a theoretical basis for his expressions, derived once again from the elastodynamic equations with harmonic variation in time.

In order to cope with scattered signals which are clearly not weak and to which first-order theory is unlikely to apply, Wesley (1965) and Aki and Chouet (1975) set up a diffusion equation for the scattered elastic energy. King et al. (1975) also found that precursors to *PP* require strong scattering. They, however, made the assumptions that only single scattering occurs and that the radiation pattern of the scattered energy is the same as for weak scattering. In other words, they assumed that the formulae derived from weak scattering theory still apply. Both of these approaches are empirical and both give a reasonable fit between theory and experiment. Unfortunately, it is not possible in either case to interpret the parameters of the model in terms of the physical properties of the medium. Nor is it known whether either is an accurate model of the elastic behaviour of a real material.

In the following sections, the various aspects of the first order theory will be drawn together into a single formula, valid for an incident wave with arbitrary variation in time. Specialisation to a harmonic input and to a statistical model is deferred, since the calculation of mean square amplitudes in the frequency domain destroys all phase (and time) information.

2. The Integral Equation

Consider an elastic material in a domain $\mathscr{D}$ with boundary $\mathscr{S}$. Its elastic parameters λ, μ (or c_{ijpq}) and density ρ are functions of position. The reference medium is defined as a similar material within a domain $\mathscr{D}^0$ with boundary $\mathscr{S}^0$, and elastic parameters λ^0, μ^0 (or c^0_{ijpq}) and density ρ^0 such that $\mathscr{S}^0$ differs from $\mathscr{S}$ within a bounded region only. Differences between the structural parameters, $\lambda^1=\lambda-\lambda^0$, $\mu^1=\mu-\mu^0$ ($\mathbf{c}^1=\mathbf{c}-\mathbf{c}^0$) and $\rho^1=\rho-\rho^0$ are also assumed to be zero outside a bounded region. It is assumed, finally, that Green's function is known for the reference medium.

Given a specific problem of wave propagation, let us suppose that the solution for the displacements in the reference medium is known and is given by $\mathbf{u}^0$. The displacements in the original structure may be written as $\mathbf{u}=\mathbf{u}^0+\mathbf{u}^1$, where $\mathbf{u}$ is understood to have an analytic continuation into any region of $\mathscr{D}^0$ exterior to $\mathscr{D}$. In general, in order to use perturbation theory, it must be assumed that $|\mathbf{u}^1|$ is everywhere small compared with the mean magnitude, or scale of $\mathbf{u}^0$.

The equation of motion in $\mathscr{D}$ is

$$\frac{\partial}{\partial x_j}\left(c_{ijpq}\frac{\partial u_p}{\partial x_q}\right)-\rho\,\ddot{u}_i=0, \tag{1}$$

where

$$\mathbf{x}=(x,y,z)=(x_1,x_2,x_3); \qquad \ddot{u}_i=\frac{\partial^2 u_i}{\partial t^2}.$$

Since $\mathbf{u}=\mathbf{u}^0$ is a solution of Equation (1) with c_{ijpq} replaced by c^0_{ijpq} and ρ replaced by ρ^0, Equation (1) may be written as

$$\frac{\partial}{\partial x_j}\left(c^0_{ijpq}\frac{\partial u^1_p}{\partial x_q}\right)-\rho^0\ddot{u}^1_i=-\frac{\partial}{\partial x_j}\left(c^1_{ijpq}\frac{\partial u_p}{\partial x_q}\right)+\rho^1\ddot{u}_i. \tag{2}$$

Let $G^j_i(\mathbf{x},\boldsymbol{\xi},t)$ be Green's function for the reference structure. Then Equation (2) may be inverted to give

$$\begin{aligned} u^1_l(\mathbf{x},t)= &-\int_{-\infty}^{\infty} dt\left[\int_{\mathscr{D}^1} dV\left\{\left[\rho^1\ddot{u}_i(\boldsymbol{\xi},\tau)-\frac{\partial}{\partial\xi_j}\left(c^1_{ijpq}\frac{\partial u_p}{\partial\xi_q}\right)\right]G^i_l(\mathbf{x},\boldsymbol{\xi},t-\tau)\right\}\right. \\ &\left.+\int_{\mathscr{S}^0} c^0_{ijpq}\frac{\partial u_p}{\partial\xi_q}G^i_l(\mathbf{x},\boldsymbol{\xi},t-\tau)\,n_j\,dS\right] \\ =&-\int_{-\infty}^{\infty} d\tau\left[\int_{\mathscr{D}^1} dV\left\{\left[\rho^1\frac{\partial u_i}{\partial\tau}\frac{\partial}{\partial t}+c^1_{ijpq}\frac{\partial u_p}{\partial\xi_q}\frac{\partial}{\partial\xi_j}\right]G^i_l(\mathbf{x},\boldsymbol{\xi},t-\tau)\right\}\right. \\ &\left.+\int_{\mathscr{S}^0} c^0_{ijpq}\frac{\partial u_p}{\partial\xi_q}G^i_l(\mathbf{x},\boldsymbol{\xi},t-\tau)\,n_j\,dS\right], \end{aligned} \tag{3}$$

where $\mathscr{D}^1$ is the part of $\mathscr{D}^0$ within which c^1_{ijpq} is non-zero and $\mathscr{S}^0$ is the reference surface, where the traction due to $\mathbf{G}$ is zero.

The effect of the heterogeneity may therefore be regarded as equivalent to that of a volume and surface distribution of forces and dipoles. The strengths of these sources depend on $\mathbf{u}$ itself, and in the following sections we consider methods by which we may represent them by approximations.

It may be noted that Equation (3) remains valid if the elastic parameters $\mathbf{c}$ and density ρ are discontinuous across given surfaces (Hudson, 1968), so long as they are piecewise continuous.

3. Slight Heterogeneity

If the deviations of the elastic properties of the material from the local average properties are small so that $\mathbf{u}^0 \simeq \mathbf{u}$ within $\mathscr{D}^1$, we may use the Born approximation (Miles, 1960) and substitute $\mathbf{u}^0$ for $\mathbf{u}$ in the volume integral in Equation (3).

The scattered field is given by the approximation

$$u^i_l(\mathbf{x},t)=-\int_{-\infty}^{\infty} d\tau\int_{\mathscr{D}}\left\{\left[\rho^1\frac{\partial u^0_i(\boldsymbol{\xi},\tau)}{\partial\tau}\frac{\partial}{\partial t}+c^1_{ijpq}\frac{\partial u^0_p}{\partial\xi_q}\frac{\partial}{\partial\xi_j}\right]G^i_l(\mathbf{x},\boldsymbol{\xi},t-\tau)\right\}dV \tag{4}$$

together with a term corresponding to scattering from perturbations from the reference surface.

Equation (4) is the basis for the majority of theoretical papers on scattering within the body of the Earth and it will be shown that all formulae for first order scattering can be put into this form.

4. Thin Scatterers

If the deviations of λ, μ and ρ from their average values are not small, the scattered field will be small only if the region in which these deviations occur is small. Thus we are led to consider small intrusions in matrix (or average) material (Herrera, 1965).

While the displacements are continuous across an interface where the material properties change, their derivatives are not. Let the normal **n** at a given point of an interface, where the material parameters change from c^m_{kjpq}, ρ^m to c^i_{kjpq}, ρ^i, be in the direction of the O_3 axis. The displacements and three components of strain are continuous; so that

$$[u_k]=0, \quad k=1,2,3$$

and (5)

$$[e_{11}]=[e_{12}]=[e_{22}]=0;$$

that is

$$e^i_{11}=e^m_{11}, \quad e^i_{22}=e^m_{22}, \quad e^i_{12}=e^m_{12},$$

where superscripts *i* and *m* refer to the intrusion and matrix respectively. Also, the tractions are continuous across the interface;

$$[\tau_{13}]=[\tau_{23}]=[\tau_{33}]=0 \tag{6}$$

that is,

$$\mu\, e^i_{13}=\mu^0\, e^m_{13}$$

$$\mu\, e^i_{23}=\mu^0\, e^m_{23}$$

and $(\lambda+2\mu)\,e^i_{33}+\lambda(e^i_{11}+e^i_{22})=(\lambda^0+2\mu^0)\,e^m_{33}+\lambda^0(e^m_{11}+e^m_{22})$.

Thus

$$e^i_{13}=\frac{\mu^0}{\mu}e^m_{13}$$

$$e^i_{23}=\frac{\mu^0}{\mu}e^m_{23}$$

$$e^i_{33}=\left(\frac{\lambda^0+2\mu^0}{\lambda+2\mu}\right)e^m_{33}-\left(\frac{\lambda^1}{\lambda+2\mu}\right)(e^m_{11}+e^m_{22}). \tag{7}$$

Alternatively,

$$c^1_{ijpq}\,e^i_{pq}=\tilde{c}_{ijpq}\,e^m_{pq} \tag{8}$$

where

$$\tilde{c}_{1111}=\tilde{c}_{2222}=(\lambda^1+2\mu^1)-(\lambda^1)^2/(\lambda+2\mu)$$

$$\tilde{c}_{3333}=(\lambda^1+2\mu^1)(\lambda^0+2\mu^0)/(\lambda+2\mu)$$

$$\tilde{c}_{1122}=\tilde{c}_{2211}=\lambda^1(\lambda^0+2\mu)/(\lambda+2\mu)$$

$$\tilde{c}_{1133}=\tilde{c}_{3311}=\tilde{c}_{2233}=\tilde{c}_{3322}=\lambda^1(\lambda^0+2\mu^0)/(\lambda+2\mu)$$

$$\tilde{c}_{1313}=\tilde{c}_{3113}=\tilde{c}_{1331}=\tilde{c}_{3131}=\mu^1\mu^0/\mu=\tilde{c}_{2323}, \quad \text{etc.}$$

$$\tilde{c}_{1212}=\tilde{c}_{2112}=\tilde{c}_{1221}=\tilde{c}_{2121}=\mu^1.$$

All other components of **c** are zero.

If the intrusion is thin compared with the length scale of variation of the displacement, the displacement field in the matrix may be approximated by the incident field $\mathbf{u}^0$;

$$\mathbf{u}^i=\mathbf{u}^m=\mathbf{u}^0, \tag{9a}$$

and similarly, for the strain field,

$$c^1_{ijpq}e^1_{pq}=c^2_{ijpq}e^m_{pq}=c^2_{ijpq}e^0_{pq}. \tag{9b}$$

where $\mathbf{c}^2$ is a tensor which reduces to $\tilde{\mathbf{c}}$ when referred to axes with O_3 along the local normal to the interface. Thus we obtain for the scattered field

$$u^1_l(\mathbf{x},t)=-\int_{-\infty}^{\infty}d\tau\int_{\mathscr{D}}dV\left\{\left[\rho^1\frac{\partial u^0_i}{\partial\tau}\frac{\partial}{\partial t}+c^2_{ijpq}\frac{\partial u^0_p}{\partial\xi_q}\frac{\partial}{\partial\xi_j}\right]G^i_l(\mathbf{x},\xi,t-\tau)\right\}. \tag{10}$$

Integration over the thickness $h(\xi)$ of the intrusion gives

$$u^1_l(\mathbf{x},t)=-\int_{-\infty}^{\infty}d\tau\int_{S}dS\left\{h(\xi)\left[\rho^1\frac{\partial u^0_i}{\partial\tau}\frac{\partial}{\partial t}+c^2_{ijpq}\frac{\partial u^0_p}{\partial\xi_q}\frac{\partial}{\partial\xi_j}\right]G^i_l(\mathbf{x},\xi,t-\tau)\right\}, \tag{11}$$

where S is the median surface of the intrusion.

Equation (10) has the same form as Equation (4). The equation may of course be applied to the problem of scattering from a distribution of inclusions, provided that the concentration of the inclusions is sufficiently dilute. It is also applicable to the problem of a slightly rough interface between two materials, in which case the roughness may be regarded as a distribution of inclusions on the interface where the two materials penetrate into each other.

It will be clear however that the parameters $\mathbf{c}^2$ cannot be calculated from Equation (8) if the inclusion is rigid ($\lambda=\mu=\infty$) or empty ($\lambda=\mu=0$). In the first case the strain within the inclusion is everywhere zero; in the second case it does not exist. The formulae derived here, based on the assumption that the strain field in the matrix is altered only slightly by the presence of the inclusion, is applicable only if the contrast between the two materials is not too large.

5. Slightly Rough Surface

The previous sections were concerned with variations in elastic properties within the material. There is now the effect of roughness at a free surface to be considered. If the reference surface $\mathscr{S}^0$ is the plane $x_3=0$, the surface integral in equation (3) may be written as

$$-\int_{-\infty}^{\infty} d\tau \int_{\mathscr{S}^0} dS\left\{c^0_{i3pq}\frac{\partial u_p}{\partial \xi_q}G^i_l\right\} = -\int_{-\infty}^{\infty} d\tau \int_{\mathscr{S}^0} dS\{\tau_{i3}G^i_l\},$$

where $\mathbf{u}$ is analytically continued, where necessary, up to the reference surface. This integral may be developed by a method due to Gilbert and Knopoff (1960). If $|f|$ is small,

$$\tau_{pq}|_{x_3=0} \simeq \tau_{pq}|_{x_3=f} - f\frac{\partial \tau_{pq}}{\partial x_3}\bigg|_{x_3=0}$$

and in the second term on the right τ_{pq} may be replaced by τ^0_{pq} to obtain a first order approximation. The free surface condition on $x_3=f$ provides a first order estimate of the first term on the right;

$$\tau_{33} - 2f_{x_1}\tau_{31} - 2f_{x_2}\tau_{32} = 0$$

and

$$l_i(f_{x_1}\tau_{11} + f_{x_2}\tau_{12} - \tau_{13}) + m_i(f_{x_2}\tau_{22} + f_{x_1}\tau_{12} - \tau_{23}) - n_i\tau_{33} = 0, \qquad i=1,2.$$

approximately where (l_i, m_i, n_i) are two vectors in the tangent plane of the surface $x_3=f(x_1,x_2)$, while the normal is $(f_{x_1}, f_{x_2}, -1)$ approximately when $f_{x_1}\left(=\frac{\partial f}{\partial x_1}\right)$ and $f_{x_2}\left(=\frac{\partial f}{\partial x_2}\right)$ are small.

Thus, to first order

$$\begin{aligned}
\tau_{33}|_{x_3=f} &= (2f_{x_1}\tau^0_{31} + 2f_{x_2}\tau^0_{32})|_{x_3=0} = 0\\
\tau_{13}|_{x_3=f} &= (f_{x_1}\tau^0_{11} + f_{x_2}\tau^0_{12})|_{x_3=0}\\
\tau_{23}|_{x_3=f} &= (f_{x_1}\tau^0_{12} + f_{x_2}\tau^0_{22})|_{x_3=0}
\end{aligned} \tag{12}$$

and the contribution to the scattered field becomes

$$\begin{aligned}
&-\int_{-\infty}^{\infty} d\tau \int_{\mathscr{S}^0} dS\left\{\left(f_{x_1}\tau^0_{11} + f_{x_2}\tau^0_{12} - f\frac{\partial \tau^0_{31}}{\partial \xi_3}\right)G^1_l\right.\\
&\qquad\left. + \left(f_{x_1}\tau^0_{12} + f_{x_2}\tau^0_{22} - f\frac{\partial \tau^0_{32}}{\partial \xi_3}\right)G^2_l - f\frac{\partial \tau^0_{33}}{\partial \xi_3}G^3_l\right\}\\
&= -\int_{-\infty}^{\infty} d\tau \int_{\mathscr{S}^0} dS\left\{\frac{\partial}{\partial \xi_1}f(\tau^0_{11}G^1_l + \tau^0_{12}G^2_l) + \frac{\partial}{\partial \xi_2}f(\tau^0_{12}G^1_l + \tau^0_{22}G^2_l)\right.\\
&\qquad\left. - f\left(\frac{\partial \tau^0_{1j}}{\partial \xi_j}G^1_l + \frac{\partial \tau^0_{2j}}{\partial \xi_j}G^2_l + \frac{\partial \tau^0_{3j}}{\partial \xi_j}G^3_l\right) - f\left(\tau^0_{1j}\frac{\partial G^1_l}{\partial \xi_j} + \tau^0_{2j}\frac{\partial G^2_l}{\partial \xi_j}\right)\right\}\\
&= \int_{-\infty}^{\infty} d\tau \int_{\mathscr{S}^0} dS\left\{f\left(\rho^0\ddot{u}^0_i G^i_l + \tau^0_{ij}\frac{\partial G^i_l}{\partial \xi_j}\right)\right\}\\
&= \int_{-\infty}^{\infty} d\tau \int_{\mathscr{S}^0} dS\left\{f(\boldsymbol{\xi})\left(\rho^0\frac{\partial u^0_i}{\partial \tau}\frac{\partial}{\partial t} + c^0_{ijpq}\frac{\partial u^0_p}{\partial \xi_q}\frac{\partial}{\partial \xi_j}\right)G^i_l(\mathbf{x}, \boldsymbol{\xi}, t-\tau)\right\},
\end{aligned} \tag{13}$$

which has the same form as Equation (11) for scattering from a thin inclusion. However, as was pointed out in the last section, Equation (13) cannot be derived from Equation (11), even though if $h(\mathbf{x})=f(\mathbf{x})$, $\rho^1=-\rho^0$, $\mathbf{c}^1=-\mathbf{c}^0$ in Equation (10), the two equations are identical.

6. The General Scattering Formula

The formulae (4), for scattering from a volume distribution of slight heterogeneity, (9), for variations in $\mathbf{c}$ and ρ within thin regions, and (13) for a rough surface or interface, all have the form

$$u_l^1(\mathbf{x},t)=-\int_{-\infty}^{\infty} d\tau \int_{\mathscr{D}} dV\left\{\left(\rho^{\dagger}\frac{\partial u_i^0}{\partial\tau}(\xi\,\tau)\frac{\partial}{\partial t}+c^{\dagger}_{ijpq}\frac{\partial u_p^0}{\partial\xi_q}\frac{\partial}{\partial\xi_j}\right)G_l^i(\mathbf{x},\xi,t-\tau)\right\}, \tag{14}$$

where $\rho^{\dagger}$ and $\mathbf{c}^{\dagger}$ are appropriate values of density and elastic parameters.

The scattered radiation in any of these cases will be the same if the volume densities of $\rho^{\dagger}$ and $\mathbf{c}^{\dagger}$ are the same. Thus, scattering from small variations in $\mathbf{c}$, with mean amplitude σ, will be of the same order of magnitude as scattering from thin intrusions with mean discontinuity Σ if the volume concentration of intrusions is approximately σ/Σ. Alternatively, thin intrusions lying on a surface layer of mean thickness h are equivalent to small variations in a layer whose thickness is of the order of $\frac{h\Sigma}{\sigma}$.

In order to obtain more information about the nature of the scattered waves, it is helpful to make some simplifying assumptions about the incident waves. This is done in the next section.

7. Ray Theory Approximation

Let us now assume that the time variation of the incident wave $\mathbf{u}^0$ is the same at each point of $\mathscr{D}^1$;

$$\mathbf{u}^0(\xi,\tau)=\mathbf{A}(\xi)f(\tau-s^0(\xi)). \tag{15}$$

Let us also assume that $\mathbf{A}$ varies slowly compared with f. This is equivalent to a representation of $\mathbf{u}^0$ as the first term of the ray expansion; $\tau=s^0(\xi)$ is the wavefront, and the ray direction is along ∇s^0.

If $\mathbf{u}^0$ represents a P wave, then

$$\nabla s^0=\frac{\mathbf{n}^0}{\alpha},\quad \mathbf{A}(\xi)=\mathbf{n}^0 A(\xi),\quad |\mathbf{n}^0|=1;$$

and if an S wave,

$$\nabla s^0=\frac{\mathbf{n}^0}{\beta},\quad \mathbf{A}(\xi)=\mathbf{m}^0 A(\xi),\quad |\mathbf{m}^0|=|\mathbf{n}^0|=1,\quad \mathbf{m}^0\cdot\mathbf{n}^0=0,$$

where α and β are the local wave speeds.

Now $G_l^i(\mathbf{x}, \xi, t-\tau)$ represents the displacement at time t in the l direction at $\mathbf{x}$ due to a point force in the i direction at ξ acting at $t=\tau$; alternatively it may be regarded as the displacement in the i direction at ξ due to a point force in the l direction at $\mathbf{x}$. For large distances,

$$G_l^i(\mathbf{x}, \xi, t-\tau) = p_i(\xi, \mathbf{x})\, B^l(\xi, \mathbf{x})\, \partial(t-\tau-s^P(\xi, \mathbf{x})) + q_i(\xi, \mathbf{x})\, C^l(\xi, \mathbf{x})\, \delta(t-\tau-s^S(\xi, \mathbf{x})) \tag{16}$$

approximately, where

$$|\mathbf{p}|=1, \qquad \nabla s^P = \mathbf{p}/\alpha$$

$$|\mathbf{q}|=1, \qquad |\beta\, \nabla s^S|=1, \qquad \mathbf{q}\cdot\nabla s^S=0.$$

7.1. Scattered *P* from an Incident *P* Wave

Under the above approximation, the scattered P wave is

$$u_l^P(\mathbf{x}, t) = -\int_{-\infty}^{\infty} d\tau \int_{\mathcal{D}} dV \{[\rho^\dagger n_i^0 p_i + c_{ijpq}^\dagger n_p^0 n_q^0 p_i p_j/\alpha^2]\, AB^l f'(\tau-s^0)\, \delta'(t-\tau-s^P)\}$$

$$= f''(t) * F_l^P(\mathbf{x}, t), \tag{17}$$

where

$$F_l^P(\mathbf{x}, t) = -\int_{\mathcal{D}} \{[p^\dagger n_i^0 p_i + c_{ijpq}^\dagger n_p^0 n_q^0 p_i p_j/\alpha^2]\, AB^l \delta(t-s^0-s^P)\}$$

$$= -\int_{S^P} AB^l \frac{(\alpha^2 \rho^\dagger n_i^0 p_i + c_{ijpq}^\dagger n_p^0 n_q^0 p_i p_j)}{\alpha\, |\mathbf{n}^0+\mathbf{p}|}\, dS, \tag{18}$$

where S^P is the surface $t=s^0+s^P$.

The smallest value of t to give a real surface S^P is $t=T^P$, the travel time from source to receiver of a P wave. At $t=T^P$, the surface S^P reduces to the curvilinear ray path with $\mathbf{n}^0+\mathbf{p}=0$ at each point.

As t increases, S^P moves out, sampling the heterogeneous medium as it goes. The speed of its advance is given by

$$v = \frac{1}{|\nabla(s^0+s^P)|} = \frac{\alpha}{|\mathbf{n}^0+\mathbf{p}|}. \tag{19}$$

Thus v is infinite at the wavefront of the scattered waves, and decreases as scattered waves arrive at the point of observation more and more from the side. For side scattering $v=\alpha/\sqrt{2}$ and the minimum value of v is $\alpha/2$ for back-scattering.

If the correlation distance of the heterogeneities (that is, roughly, the spatial wavelength of the variations in $\mathbf{c}$ and ρ) is a, the correlation time of the scattered signal is approximately a/v.

Thus, the scattered wave begins with short correlation times (high frequencies) corresponding to in-line or forward scattering, and later settles down to vibrations centred on frequencies directly proportional to the size of the scatterers (side and back scattering). This is in accord with earlier findings (Knopoff and Hudson, 1964) that scattering at high frequencies is confined to ray paths close to the incident path, while low frequency scattering should be observed arriving from all possible directions.

The main difference between the formula (18) and the corresponding expression for acoustic scattering lies in the radiation pattern for each element of the variable medium regarded as a radiating source, given by the expression

$$\alpha^2 \rho^\dagger n_i^0 p_i + c_{ijpq}^\dagger n_p^0 n_q^0 p_i p_j.$$

Scattering from variations in density is governed by the angular variation of $\mathbf{n}^0 \cdot \mathbf{p}$ ($=\cos\chi$, say). Also, if the elastic properties are isotropic

$$c_{ijpq}^\dagger n_q^0 n_p^0 p_i p_j = \lambda^\dagger + 2\mu^\dagger (\mathbf{n}^0 \cdot \mathbf{p})^2 = \kappa^\dagger + \frac{4\mu^\dagger}{3} P_2(\cos\chi), \tag{20}$$

where $\kappa^\dagger$ is the corresponding bulk modulus.

Thus, variations in the bulk modulus give rise to an isotropic radiation pattern while variations in rigidity give rise to a radiation governed by $P_2(\cos x)$ the Legendre polynomial of order 2. Variations in density, as shown above, give rise to a radiation pattern according to $P_1(\cos\chi)$. The angle $(\pi - \chi)$ is the angle of scattering.

These radiation patterns were first noted by Miles (1960).

7.2. Scattered S from an Incident P Wave

Substitution of the S wave component of Green's function, gives the scattered S wave:

$$u_l^S(\mathbf{x}, t) = -\int_{-\infty}^{\infty} d\tau \int_{\mathscr{D}} dV \{(\rho^\dagger n_i^0 q_i + c_{ijpq}^\dagger n_p^0 n_q^0 q_i p_j/\alpha\beta) A C^l$$
$$\times f'(\tau - s^0)\,\delta'(t - \tau - s^S)\},$$

where, here, $\mathbf{p} = \beta \nabla s^S$.

So

$$u_l^S(\mathbf{x}, t) = f''(t) * F_l^S(\mathbf{x}, t), \tag{21}$$

where

$$F_l^S(\mathbf{x}, t) = -\int_{S^S} A C^l \frac{(\alpha\beta\rho^\dagger n_i^0 q_i + c_{ijpq}^\dagger n_p^0 n_q^0 q_i p_j)}{|\beta\mathbf{n}^0 + \alpha\mathbf{p}|} dS \tag{22}$$

and S^S is the surface $t = s^0 + s^S$.

This surface starts at the point of observation, at $t = T^P$ again. However, $\beta\,\mathbf{n}^0 + \alpha\,\mathbf{p}$ is not zero there, and so scattered wave amplitudes from the surface integral will be small to begin with.

As t increases, S^S moves out with speed

$$v=\frac{1}{|\nabla(s^0+s^S)|}=\frac{\alpha\beta}{|\beta\,\mathbf{n}^0+\alpha\,\mathbf{p}|}. \tag{23}$$

This is largest for forward scattering $\left(v=\frac{\beta}{1-\beta/\alpha}\right)$ but it does not vary nearly as much as the corresponding quantity for scattered P, and in-line scattering will have slightly higher frequencies than side or back scattering. Again, the correlation time is approximately a/v when the correlation distance of the scatterers is a.

There is no very high frequency scattered signal in this case; that is, at high frequencies the conversion from P to S is small (Knopoff and Hudson, 1967).

The radiation pattern for isotropic heterogeneity is given by

$$\alpha\beta\rho^\dagger\,\mathbf{n}^0\cdot\mathbf{q}+2\mu^\dagger(\mathbf{n}^0\cdot\mathbf{p})(\mathbf{n}^0\cdot\mathbf{q}); \tag{24}$$

that is, a cosine pattern for $\rho^\dagger$ and a double cosine pattern for $\mu^\dagger$.

If the scattering region is far from the receiver point, there will be a time delay from $t=T^P$ until the surface S^S grows large enough to intersect the scattering region. This time will correspond to a minimum time path for an incident ray to get to a scattering point (at speed α) and then travel by a shear ray path to the receiver (at speed β).

It has sometimes been stated that, if the wavelength of the incident wave is sufficiently short, conversion from one wave-type to the other ceases and that the acoustic equations may be used. However the acoustic equations do not give the correct radiation patterns.

8. Results for a Homogeneous Reference Medium

In order to illustrate these results more clearly, the expressions will be evaluated for the case when the reference medium is homogeneous and isotropic. Green's function is now

$$G_i^j(\mathbf{x},\xi,t)=\frac{p_i\,p_j}{4\pi\,p^0\,R\,\alpha^2}\,\delta(t-R/\alpha)+\frac{(\delta_{ij}-p_i\,p_j)}{4\pi\,p^0\,R\,\beta^2}\,\delta(t-R/\beta) \\ +\text{lower order terms} \tag{25}$$

where

$R=|\mathbf{x}-\xi|$ and $\mathbf{p}=(\xi-\mathbf{x})/R$.

Thus

$$B^l = \frac{p_l}{4\pi\rho^0 R\alpha^2}, \qquad s^P = R/\alpha$$

and

$$C^l q_i = \frac{\delta_{il} - p_i p_l}{4\pi\rho^0 R\beta^2}, \qquad s^S = R/\beta.$$

If the incident wave is a uniform plane wave;

$$\mathbf{u}^0(\boldsymbol{\xi}, \tau) = A\,\mathbf{n}^0 f(\tau - \mathbf{n}^0 \cdot \boldsymbol{\xi}/\alpha) \tag{26}$$

with A and $\mathbf{n}^0$ constant. The scattered P-wave is given by Equation (17) with

$$F_l^P(\mathbf{x}, t) = \frac{-A}{4\pi\rho^0\alpha^3} \int_{S^P} \frac{p_l}{R} \frac{(\alpha^2\rho^\dagger n_i^0 p_i + c_{ijpq}^\dagger n_p^0 n_q^0 p_i\, p_j)}{|\mathbf{n}^0 + \mathbf{p}|} dS. \tag{27}$$

The surface S^P is given by

$$t = s^0 + s^P$$

i.e.

$$t = \mathbf{n}^0 \cdot \boldsymbol{\xi}/\alpha + R/\alpha.$$

Let us take the origin of Cartesian axes at the point of observation with the O_3 axis along $\mathbf{n}^0$; the equation for S^P becomes

$$\alpha t = \xi_3 + R. \tag{28}$$

This is a paraboloid of revolution with focus at the origin (the point of observation) and directrix at $\xi_3 = \alpha t$.

The surface moves out with speed $v = \dfrac{\alpha}{\sqrt{2}(1 + \xi_3/R)^{\frac{1}{2}}}$.

With cylindrical polar coordinates (r, θ, ζ) having origin at $\boldsymbol{\xi} = 0$ and $\xi_3 = \zeta$, we obtain

$$F_l^P(\mathbf{x}, t) = \frac{-A}{4\pi\rho^0\alpha^3} \int_{S^P} p_l(\alpha^2\rho^\dagger n_i^0 p_i + c_{ijpq}^\dagger n_p^0 n_q^0 p_i\, p_j)\, d\zeta\, d\theta. \tag{29}$$

The scattered S waves are given by

$$F_l^S(\mathbf{x}, t) = \frac{-A}{4\pi\rho^0\beta^2} \int_{S^S} \frac{(\alpha\beta\rho^\dagger n_i^0 + c_{ijpq}^\dagger n_p^0 n_q^0 p_j)(\delta_{il} - p_i p_l)}{|\beta\mathbf{n}^0 + \alpha\mathbf{p}|} \frac{dS}{R}. \tag{30}$$

The surface S^S is

$$t = s^0 + s^S$$

i.e.

$$t=\xi_3/\alpha+R/\beta \tag{31}$$

or

$$R=\frac{\beta}{\alpha}(\alpha t-\xi_3),$$

which is an ellipse with the origin as focus, $\xi_3=\alpha t$ as directrix, and with ellipticity β/α. Finally, in terms of cylindrical coordinates,

$$F_l^S(\mathbf{x},t)=\frac{-A}{4\pi\rho^0\beta^2\alpha}\int_{S^S}(\alpha\beta\rho^\dagger n_i^0+c_{ijpq}^\dagger n_p^0 n_q^0 p_j)(\delta_{il}-p_i p_l)\,d\theta\,d\zeta. \tag{32}$$

The amplitudes of the scattered waves depend on the shape of the scattering region and on the radiation pattern of each elementary scatterer. As time increases the surface S^P or S^S moves outward through the medium; in-line scattering gives way to side and back-scattering and the envelope of the scattered signal changes accordingly. However, it is noticeable that there is no obvious decay due to geometrical spreading in the formulae (28) and (31). This is because the geometrical spreading of individual ray tubes is balanced by the steadily increasing size of the surface S^P or S^S.

In a model in which the scattering region is contained within an infinite plane-sided layer, the theoretical scattered signal will have infinite duration. This shows that scattered signals may well be very extended in time.

9. Statistical Properties of Scattered Signals

According to Equations (17) and (21), the scattered waves are given by the convolution of the time function of the incident acceleration with either $\mathbf{F}^P$ or $\mathbf{F}^S$. Expressions (29) and (32) for $\mathbf{F}^P$ and $\mathbf{F}^S$ show that both are linearly related to $\rho^\dagger$ and $\mathbf{c}^\dagger$, and therefore their statistical properties are easily obtained from the statistical properties of $\rho^\dagger$ and $\mathbf{c}^\dagger$.

Rather than regard the scattering region as a single sample of an infinite ensemble, it is better for many purposes to derive correlations of the signal in time from correlations of the material properties in space. This approach has already been introduced in the connection of the correlation length a of the material variations with the correlation time a/v of the signal, where v is the speed of advance of the surface S^P or S^S.

Similarly mean square amplitudes and correlations of amplitudes at two different points can be calculated. The advantage of this method over earlier investigations is that it operates in the time domain, in which the observations are made, and the statistical interpretation has more direct application to the data. A mean value, in this case, is an average over a restricted length of signal, or region of space.

10. Applications of the First-Order Theory

If the above formulae are to be used, the energy converted by scattering must be small compared with energy in the incident wave. However, if this is so, it might be expected that the scattering will be seen as low amplitude noise on a seismic trace, and ignored. Important applications of the hypothesis of scattering from heterogeneity in the Earth, however, occur only when the supposed scattering is comparable with the known standard phases on a seismogram. In this case, first-order theory may not in general be applied with any confidence.

There are situations, however, where weak amplitude scattering theory may be applicable; these include cases where the primary signal has been obscured and the scattered signal, by travelling by a separate path, is at full strength (see, for example, Douglas, 1973). Three examples may be noted.

Firstly, precursors to *PKIKP* are observed in the shadow region of *PKP*, the primary signal from which the precursors are assumed to have been scattered. The precursors arrive at a time when the record is quiet and can be clearly seen for this reason. Thus, in the first instance, one may have good reason to hope that first-order theory will explain the phenomenon. Recent studies (see, for instance, Doornbos, 1976) have suggested that, in fact, the heterogeneity producing the precursors is not small. If this is so, a new theory for multiple scattering needs to be set up.

A second example is the study of precursors to *PP* at epicentral distances greater than about 100°. The primary wave in this case is *P*, which is obscured at these distances by the core. Scattered signals would travel by paths entirely in the crust and mantle and would be expected to be relatively large in amplitude, as seen on the seismogram.

Finally *S* to *P* scattering travels ahead of the primary *S* wave and appears in the coda of *P*. If the direction of radiation is near a node of *P* and an antinode of *S*, the first arrival will be relatively small and will be followed by scattering from a relatively large *S* wave. In this way an extended and large amplitude coda might be produced. Such a mechanism may well be the explanation of the occurrence of both simple and complex signals derived from similar (simple) earthquake sources and travelling along nearly the same source-receiver path. The clue lies in the orientation of the source.

Other mechanisms have been explored by which the primary wave may be obscured, such as by attenuation of the primary wave by regions of high damping (Douglas, 1973), or by anomalous geometrical dispersion (Davies and Julian, 1972).

The above argument does not imply that scattering is small in these examples, but rather that these are situations where one may reasonably begin the investigation with a theory based on a first-order perturbation.

The difficulty with the hypothesis of strong scattering is that, in the absence of a satisfactory theory, it is difficult to check. However, if extended regions of strong scattering exist in the Earth, the problem would be to explain the existence of simple signals – signals which have not apparently been degraded by a scattering process.

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