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Energetics of the Earth's Core

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Abstract. The energy supplied to generate the earth's magnetic field must ultimately result in heat flowing across the core-mantle boundary and through the earth's surface. If the liquid core is stirred by thermal convection then only a small fraction of the total heat is dissipated in the electric currents, and in order to explain the observed field at least 10^{11} watt and probably 10^{13} watt of the earth's surface heat flux must originate deep inside the core. If the core is cooling and there is concomitant chemical differentiation, a large amount of gravitational energy is released. This energy, unlike the heat released, is completely dissipated in the electric currents and enables the same magnetic field to be generated with a much lower heat flux. Chemical differentiation is therefore favoured as the energy source for the dynamo. The importance of gravitational settling depends on the density jump at the inner core boundary and on the stratification parameter in the outer core, both of which can, in principle, be determined seismologically.

Key words: Earth's core – Dynamo – Energetics.

Introduction

Gubbins (1976, paper I) has investigated the energetics of generating the earth's magnetic field by a dynamo process driven by thermal convection, and found that a lower bound of 10^{11} watt of heat is required. This heat flows across the core mantle boundary and is either carried away by deep mantle convection, thermal conduction being too slow, or else accumulates at the bottom of the mantle. The magnetic field has persisted for 3 Gy and in the latter case the mantle would become very hot. It is therefore assumed that convection carries this heat to the surface where it forms part of the observed surface heat flux. The lower bound of 10^{11} watt is found by choosing the poorly known parameters so as to minimise the heat flux, and the true value would undoubtedly have to be much larger. For example at least 500 times as much heat is needed to sustain a toroidal field of 100 Gauss inside the core. The radial distribution of heat sources is also important because the dynamo is most efficient when they are located in the inner core. In all, one might expect 10^{13} watt or more to reach the earth's surface from the core, but this is almost all of the observed heat flux. Recent estimates of heat flow are higher than earlier values (Williams and von Herzen, 1974), mainly because of the effects of hydrothermal circulation, but even so the radiogenic materials are concentrated in the crust and 10^{13} watt seems a reasonable upper limit for the internally generated heat. Evidence from minerals such as Harzburgite and Kimberlite suggests that the mantle has an appreciable radioactive content, and there does not seem to be enough heat left to drive the dynamo by thermal convection. Before looking for alternative mechanisms for the dynamo, consider the question: are there any signs in the observations that the heat has come from the core? Rotation exerts a strong influence on core convection and there will be a characteristic lattitude dependence (Busse, 1970). This lattitude dependence will have persisted for the last 3Gy and will influence mantle convection and possibly the tectonics at the surface. Therefore in searching for this lattitude dependence of the heat flux, only the radioactive heating should be subtracted from the observations. Chapman and Pollack (1975) have produced global heat flow maps derived from observations and from observations supplemented by predictions based on the major tectonic provinces. There is some suggestion, in these maps, of low heat flow near the poles but the shortage of any data in the polar regions and the southern hemisphere makes speculation risky. In any case the anomalies are small. This is further evidence that the core contribution to the heat flux is well below that required for the thermally convecting dynamo, and in the rest of this paper an alternative driving mechanism is sought, namely release of gravitational energy.

The Energy Source

Verhoogen (1961) has shown that energy for the magnetic field can come from cooling of the Earth and gradual crystallization of the inner core. Some seismic models feature a density jump at the inner core boundary, for example 0.8 or 0.6 Mg m^{-3} in the models 1066A and 1066B of Gilbert and Dziewonski (1975). As the inner core grows by accreting denser material gravitational energy is lost, some of which is available to do work in generating magnetic field. Metals do not change volume much on melting, so the density jump, if one exists at all, must be due to a compositional change. Furthermore, Hugoniot data suggests that the density of the outer core is too small for pure iron or iron-nickel alloy, and lighter elements, notably silicon or sulphur, are believed to be present. On freezing, a heavier component with more iron can separate out, enriching the liquid in the lighter component until a eutectic is reached, details of the process depending on the phase diagram. The problem was formulated by Braginsky (1964) for an Fe-Si core, but he did not consider the gross thermodynamics of the problem which leads to an instructive estimate of dynamo "efficiency".

The local expression of the first law of thermodynamics can be written as:

$$\frac{\partial}{\partial t}\left(\rho e+\tfrac{1}{2}\rho \mathbf{v}^2+\frac{\mathbf{B}^2}{2\mu_0}\right)=-\operatorname{div}\left[\rho \mathbf{v}\left(\tfrac{1}{2}\mathbf{v}^2+e+\frac{p}{\rho}\right)+\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}-\mathbf{v}\cdot\boldsymbol{\tau}'-k\nabla T\right]$$
$$+H+\rho\mathbf{v}\cdot\nabla\psi \tag{1}$$

where ρ is density, $\mathbf{v}$ the velocity, $\mathbf{B}$ and $\mathbf{E}$ the magnetic and electric fields, e the internal energy, p the pressure, $\boldsymbol{\tau}'$ the deviatoric stress, k the thermal conductivity, T the temperature, H the local heat generation by radioactive sources, and ψ the gravitational potential such that $\mathbf{g}=\nabla\psi$ where $\mathbf{g}$ is acceleration due to gravity. This expression agrees with the integral form in Backus (1975), but the equation in Hewitt et al. (1975) applies only when the gravitational potential is independent of time and generated from external sources, and it differs from (1) by the term $\rho\frac{\partial\psi}{\partial t}$. We allow the core to evolve with time and consider the gross energy balance by integrating (1) over V, the volume occupied by the core. The core can contract, so at the boundary $\mathbf{v}\cdot\mathbf{n}=\mathscr{V}$ where $\mathbf{n}$ is the outward pointing unit normal and $\mathscr{V}$ is constant. The deviatoric stress integral is zero for both stress-free and no-slip boundary conditions, thereby excluding the driving force for the precessional dynamo. The space outside V is an electrical insulator and contains no mass generating gravitational forces inside V. Using the equation of conservation of mass and Reynolds transport theorem, equation (1) gives

$$Q=-\oint_S k\nabla T\cdot\mathbf{dS}+R+\Sigma+G+P \tag{2}$$

where

$$R=\int_V H\,\mathrm{dV},$$

$$\Sigma=-\frac{\mathrm{d}}{\mathrm{dt}}\int_V \rho e\,\mathrm{dV}=\int_V \sigma\,\mathrm{dV},$$

$$G=\int_V \rho\mathbf{v}\cdot\nabla\psi\,\mathrm{dV},$$

$$P=-\oint_S p\mathbf{v}\cdot\mathbf{dS},$$

$$\sigma=-\rho\frac{\mathrm{De}}{\mathrm{Dt}}.$$

Terms in the rate of change of kinetic energy inside V, $\frac{\mathrm{d}}{\mathrm{dt}}\int_V \tfrac{1}{2}\rho\mathbf{v}^2\,\mathrm{dV}$, and magnetic energy $\frac{\mathrm{d}}{\mathrm{dt}}\int\frac{\mathbf{B}^2}{2\mu_0}\mathrm{dV}$ have been omitted from (2). Rough estimates of the kinetic and magnetic energies are 10^{16} and 10^{21} joules, respectively, so that very rapid changes would have to occur for these terms to be significant in (2). In particular, the energy change due to a gradual decay of the magnetic field by a factor of ten or so since the Precambrian is not significant.

(2) expresses the statement that heat flowing out of the core is equal to the sum of radioactive heating, the rate of loss of internal energy, the rate of loss of gravitational energy and the work done by pressure on the surface. This last term must come from a change in the gravitational energy of the mantle.

All the terms on the right hand side of (2) can be estimated for the core. The radioactive contribution, R, lies between zero and 10^{13} watt depending on whether one believes that the isotope K^{40} is present. Taking Verhoogen's (1961) calculation as a basis, the energy released by cooling is about 10^{11} watt, assuming that the core cools by 10–45 °K over 3 Gy. The heat seen at the surface would also contain that lost by a cooling mantle. Taking the specific heat of mantle material as 10^3 joules $(\mathrm{kg})^{-1}$ $(\mathrm{K}°)^{-1}$ and a mean density of 5 Mg m^{-3} then the heat released is $5 \cdot 10^{28}$ joules or $5 \cdot 10^{11}$ watt for 3 Gy. This agravates the heat flux problem at the surface. The gravitational energy may be estimated roughly as follows. Using the equation of continuity we have:

$$\int_V \rho \mathbf{v} \cdot \nabla \psi \, \mathrm{dV} = \oint_S \rho \mathbf{v} \psi \cdot \mathbf{dS} + \int_V \psi \frac{\partial \rho}{\partial t} \, \mathrm{dV}. \tag{3}$$

Consider a simple rearrangement of material which entails no changes in volume, so that $\mathbf{v} \cdot \mathbf{n} = 0$ on S. The total mass remains constant so that:

$$\int_V \frac{\partial \rho}{\partial t} \, \mathrm{dV} = 0$$

and an estimate of $\int_V \psi \frac{\partial \rho}{\partial t} \, \mathrm{dV}$ is unaffected by the choice of reference for ψ. Take ψ to be zero at the core mantle boundary. Using the parameters from Verhoogen's calculation, 10^{11} watt is released by freezing with a latent heat of 4×10^5 joule kg^{-1}, and therefore 25 m^3 of material is frozen in one second. Suppose that the extra light component released is distributed uniformly throughout the outer core, lowering its density by $\frac{25 \Delta \rho}{V_c}$ where V_C is the volume of the outer core and $\Delta \rho$ the density jump at the inner core boundary in Mg m^{-3}.

Taking parameters from the seismic model 1066 A (Gilbert and Dziewonski, 1975) we can calculate ψ throughout the outer core by integrating $\mathbf{g}$ numerically, and find that:

$$\int_V \psi \frac{\partial \rho}{\partial t} \, \mathrm{dV} = \Delta \rho \times 2 \times 10^{11} \text{ watt} = \int_V \rho \mathbf{v} \cdot \nabla \psi \, \mathrm{dV}$$

where $\Delta \rho$ is positive for a higher density in the inner core. For model 1066 A we have $\Delta \rho = 0 \cdot 87$ Mg m^{-3}, and so the contribution to G is 1.7×10^{11} watt. Extra energy is released by thermal contraction but it will be shown later that this does not play a direct part in generating magnetic field. A similar calculation, including the surface integral, gives the energy released assuming the density jump is due to melting. More energy is released, the estimate being over 10^{12} watt as reported in I.

The work done by pressure forces on the surface is:

$$P = -\oint_S p\mathbf{v}\cdot\mathbf{dS}.$$

If the core cools at 10 °C per Gy then with a thermal expansion coefficient of $4\times10^{-6}\,(\mathrm{K}^\circ)^{-1}$ the rate of reduction of the core radius is about $10^{-15}\,\mathrm{ms}^{-1}$ (The cooling does not lead to an observable change in earth radius or moment of inertia). Taking this value for $-\mathscr{V}$ and a pressure of 1 Mb or $10^{11}\,\mathrm{Nm}^{-2}$, P can be estimated as $-p\mathscr{V}\times4\pi r_c^2$, where r_c is the core radius, or about 2×10^{10} watt. All the terms in equation (2) could be of comparable magnitude and must be retained in the energy budget. However, the magnetic field does not enter into the energy balance, and to obtain information about the dynamo the entropy must be considered.

"Efficiency"

Hewitt et al. (1975) and Backus (1975) showed that magnetic dissipation, Φ, could be calculated from the entropy equation and they placed upper bounds on the "efficiency" of a dynamo in steady state:

$$\frac{\Phi}{Q} \leq \frac{T_{\max} - T_{\min}}{T_{\min}} \tag{4}$$

where $T_{\max}$, $T_{\min}$ are the maximum and minimum temperatures within the volume. Their argument is not quite right for radioactive heating because the system is not in steady state. The number of radioactive atoms and therefore the entropy changes with time. This entropy change is presumed small. The present calculation is not steady state because the Earth gradually cools, and we must estimate the changing entropy.

The entropy equation in Hewitt et al. is valid for any gravitational potential:

$$\rho\frac{\mathrm{Ds}}{\mathrm{Dt}} = \frac{\nabla\cdot(k\nabla T)}{T} + \frac{(H+\phi)}{T} \tag{5}$$

where ρs, ϕ are the entropy and dissipation per unit volume. Viscous dissipation is negligible in the core so that

$$\phi = \frac{\mathbf{J}^2}{\lambda}$$

where λ is the electrical conductivity and $\mathbf{J}$ the electric current vector. To calculate the entropy change we use:

$$T\,\mathrm{ds} = de - \frac{p\,\mathrm{d}\rho}{\rho^2}.$$

After some algebra and using the continuity equation, (5) becomes

$$\frac{-\sigma+p\nabla\cdot\mathbf{v}}{T}=\frac{\nabla\cdot(k\nabla T)}{T}+\frac{(H+\phi)}{T}. \tag{6}$$

Note that by multiplying (6) by T and integrating over the whole core, and using (2) we get:

$$\Phi=\int_V \phi\,\mathrm{dV}=-\int_V (\mathbf{v}\cdot\nabla p+\mathbf{v}\cdot\rho\nabla\psi)\,\mathrm{dV}. \tag{7}$$

This result is obtained more easily by forming the scalar product of the equation of motion with $\mathbf{v}$ and integrating. It represents the statement that the dissipation equals the work done by the pressure gradient plus the work done by gravitational forces. In an incompressible fluid with no thermal expansion, $\nabla\cdot\mathbf{v}=0$ and $\mathbf{v}\cdot\mathbf{n}=0$ on the bounding surface. Then

$$\int_V \mathbf{v}\cdot\nabla p\,\mathrm{dV}=\oint_S p\mathbf{v}\cdot\mathbf{dS}-\int_V p\nabla\cdot\mathbf{v}\,\mathrm{dV}=0$$

and (7) gives:

$$\Phi=G.$$

Thus if we have no compression or thermal convection, all the gravitational energy released is converted into heat by magnetic dissipation. This is the essential difference between gravitationally and thermally driven dynamos.

To deal with the general problem we return to (6) and integrate over the whole core. Rearranging gives:

$$\int_V \frac{H+\phi}{T}\,\mathrm{dV}+\int_V \frac{\sigma-p\nabla\cdot\mathbf{v}}{T}\,\mathrm{dV}=-\oint_S \frac{k\nabla T\cdot\mathbf{dS}}{T}-\int_V k\left(\frac{\nabla T}{T}\right)^2\mathrm{dV}. \tag{8}$$

The term in $\sigma-p\nabla\cdot\mathbf{v}$ is the rate of change of entropy and is zero in steady state. It can therefore be estimated from the gradual cooling and contraction of the Earth as described in the last section. The problem is idealised to a steady state system superimposed on a gradually cooling earth. This implies a time averaging of the real situation to eliminate fluctuating effects. Let $\mathbf{u}$ denote the velocity of slow contraction, and p', T' the averaged pressure and temperature. Then:

$$\int_V \frac{\sigma-p\nabla\cdot\mathbf{v}}{T}\,\mathrm{dV}=\int_V \frac{\eta_1+\eta_2}{T'}\,\mathrm{dV}$$

where

$$\eta_1=-\rho\frac{\partial e}{\partial t}-\rho\mathbf{u}\cdot\nabla e,$$

$$\eta_2=p'\nabla\cdot\mathbf{u}.$$

The distinction between the η's and $\sigma-p\nabla\cdot\mathbf{v}$ is very important because the individual integrals $\int_V \frac{\sigma}{T}\mathrm{dV}$ and $\int_V \frac{p\nabla\cdot\mathbf{v}}{T}\,\mathrm{dV}$ are not zero in steady state, although their contributions exactly cancel. Also if we could estimate $\int_V p\nabla\cdot\mathbf{v}\,\mathrm{dV}$ just from the gradual contraction then (7) would give the dissipation directly, but this calculation would be grossly wrong.

η_2 is everywhere positive, and suppose for the moment that η_1 is also positive everywhere. Setting $T'=T$ we have:

$$\int_V \frac{\eta_1+\eta_2}{T}\,\mathrm{dV} \geqq \frac{\int_V(\eta_1+\eta_2)\,\mathrm{dV}}{T_{\max}} = \frac{\Sigma+P_1}{T_{\max}}$$

where $P_1 = -\int_V p\nabla\cdot\mathbf{u}\,\mathrm{dV}$.

If the core surface is held at the tmperature $T_{\min}$, then (8) gives, using (2) again:

$$\frac{\Sigma+P_1+R+\Phi}{T_{\max}} \leqq \frac{\Sigma+R+P+G}{T_{\min}}$$

or:

$$\frac{\Phi-(G+P-P_1)}{Q} \leqq \frac{T_{\max}-T_{\min}}{T_{\min}}. \tag{9}$$

(9) expresses the statement that the magnetic dissipation is equal to the gravitational energy plus the work done by pressure forces on the surface minus the work done in contraction, plus a fraction of the heat flux, Q, which cannot exceed $\frac{T_{\max}-T_{\min}}{T_{\min}}$. Neglecting compressional terms and setting $G=0$ gives the result obtained by Hewitt et al. (1975) and Backus (1975), so that the dynamo driven by cooling has the same efficiency as that driven by radioactive heating.

The compressional term P_1-P is:

$$P_1-P=\int_V[-\nabla\cdot(p'\mathbf{u})+p'\nabla\cdot\mathbf{u}]\,\mathrm{dV} = -\int_V \mathbf{u}\cdot\nabla p'\,\mathrm{dV}.$$

This is the work done against the pressure forces in the slow contraction. The problem of a cooling, self gravitating body was studied by Lapwood (1952), who considered a "quasi-hydrostatic" contraction. With this assumption

$$\nabla p' = \rho\nabla\psi$$

and

$$P_1-P=-\int_V \rho\,\mathbf{u}\cdot\nabla\psi\,\mathrm{dV}.$$

This is the work done by gravitational forces in the slow contraction and leads to a contribution to the changing gravitational energy. (9) shows that this part of the gravitational energy is not dissipated in the electric currents, but instead part appears directly as heat (P) and part goes into work done during contraction

(P_1). The remaining gravitational energy release comes from rearrangement of matter. The estimate of 1.7×10^{11} watt for gravitational energy calculated in the previous section was an estimate of $(G+P-P_1)$ because it was derived from rearrangement of matter. All of this energy is dissipated in the magnetic field. In the estimate based on freezing in paper I, the contribution $P_1 - P$ must be subtracted to give the energy directly available.

Lastly, consider the case when η_1, the local rate of change of internal energy, is negative. This would be the case if gravitational rearrangement were taking place without cooling or change in volume, because:

$$\eta_1 = -\frac{\partial}{\partial t}(\rho\, e) = -\frac{\partial \rho}{\partial t}\, e$$

and $\frac{\partial \rho}{\partial t}$ would be positive in lower regions but negative in upper regions. Let V_1 be the region in which η_1 is positive and V_2 be where η_1 is negative. Then:

$$\int_V \frac{\eta_1}{T}\,\mathrm{dV} \geqq \int_{V_1} \eta_1\,\mathrm{dV} + \frac{\int_{V_2} \eta_1\,\mathrm{dV}}{T_{\mathrm{min}}}.$$

(9) becomes:

$$\frac{\Phi-(G+P-P_1)}{Q'} \leqq \frac{T_{\mathrm{max}} - T_{\mathrm{min}}}{T_{\mathrm{min}}} \tag{10}$$

where

$$Q' = R + G + P + \int_{V_1} \eta_1\,\mathrm{dV}.$$

Q' is greater than Q and the "efficiency" in this case is potentially greater than in (9).

Diffusion of Matter and the Heat of Reaction

Braginsky (1964) has treated the general problem of two-component diffusion in the core, with reference to an iron-silicon mixture, including the effects of a heat of reaction between the components. In this case there are extra contributions to the entropy balance. Following Landau and Lifshitz (1959, sections 57–58), define the concentration, c, to be the ratio of the mass of one component to the mass of the whole fluid, and **i**, the mass flux of one component transported by diffusion through unit area in unit time. Then the continuity equation for one component is:

$$\rho\,\frac{\mathrm{Dc}}{\mathrm{Dt}} = -\operatorname{div}\mathbf{i}. \tag{11}$$

The energy equation is unchanged except that the heat flux vector, $\mathbf{q}$, now depends on the concentration gradient as well as the temperature gradient. The internal energy now contains effects of the two component mixture. Introduce the chemical potential μ:

$$de = T\,\mathrm{ds} - p\,\mathrm{dV} + \mu\, dc.$$

Integrating the energy equation over V gives, as before

$$\oint_S \mathbf{q}\cdot\mathbf{dS} = R + \Sigma + G + P. \tag{12}$$

The entropy equation is (Landau and Lifshitz, 1959):

$$\rho T \frac{\mathrm{Ds}}{\mathrm{Dt}} = -\nabla\cdot\mathbf{q} + H + \phi + \mu\,\mathrm{div}\,\mathbf{i}. \tag{13}$$

The heat flux, $\mathbf{q}$, depends on the concentration gradient as well as the temperature gradients, and in general the flux $\mathbf{i}$ will depend on temperature, concentration and pressure gradients. Using the Onsager reciprocal relations (Landau and Lifshitz, 1959; Malvern, 1969) gives the phenomenological relations:

$$\mathbf{q} - \mu\,\mathbf{i} = -k\nabla T + \frac{\beta T}{\alpha}\,\mathbf{i}, \tag{14}$$

$$\mathbf{i} = -\alpha\nabla\mu - \beta\nabla T$$

where increase of entropy ensures that $k, \alpha > 0$ but β can take either sign. Divide (13) by T and integrate over V. The integral in $\mathbf{q}$ is expanded as follows.

$$\int_V \frac{\nabla\cdot\mathbf{q}}{T}\,\mathrm{dV} = \int_V \frac{\nabla\cdot(\mathbf{q}-\mu\,\mathbf{i})}{T}\,\mathrm{dV} + \int_V \frac{\nabla\cdot(\mu\,\mathbf{i})}{T}\,\mathrm{dV}$$

$$= \frac{\oint_S (\mathbf{q}-\mu\,\mathbf{i})\cdot\mathbf{dS}}{T_{\min}} + \int_V \frac{(\mathbf{q}-\mu\,\mathbf{i})\cdot\nabla T}{T^2}\,\mathrm{dV} + \int_V \frac{\nabla\cdot(\mu\,\mathbf{i})}{T}\,\mathrm{dV}.$$

Using (14), (13) becomes:

$$\int_V \rho\frac{\mathrm{Ds}}{\mathrm{Dt}}\,\mathrm{dV} = -\frac{Q}{T_{\min}} + \int_V \left[k\left(\frac{\nabla T}{T}\right)^2 + \frac{\mathbf{i}^2}{\alpha T}\right]\mathrm{dV} + \int_V \frac{H+\phi}{T}\,\mathrm{dV}.$$

The argument of the previous section is used to replace the left hand side by terms depending only on the slow evolution of the Earth. Then:

$$\int_V \rho\frac{\mathrm{Ds}}{\mathrm{Dt}} = \int_V \frac{-\sigma + p\nabla\cdot\mathbf{v} + \mu\nabla\cdot\mathbf{i}}{T}\,\mathrm{dV}$$

giving:

$$\int_V \frac{\sigma - p\nabla\cdot\mathbf{v} - \mu\nabla\cdot\mathbf{i} + H + \phi}{T}\,\mathrm{dV} = \frac{Q}{T_{\min}} - \int_V \left[k\left(\frac{\nabla T}{T}\right)^2 + \frac{\mathbf{i}^2}{\alpha T}\right]\mathrm{dV}.$$

If the integrand on the left hand side is always positive then there is the simple inequality:

$$\frac{\Phi-(G+P-P_1-C)}{Q} \leqq \frac{T_{\max}-T_{\min}}{T_{\min}}$$

where $C=\int_V \mu \nabla \cdot \mathbf{i}\,\mathrm{dV}$. The term $\frac{i^2}{\alpha T}$ is the entropy of diffusion of material, analogous to the entropy of conduction of heat. The integral C may be written as $-\int_V \mathbf{i} \cdot \nabla \mu\,\mathrm{dV}$ and gives the energy lost by a flux of material along a gradient in chemical potential.

Assigning numerical values to the chemical constants is very difficult because of the high pressure and uncertainty in composition. Braginsky (1964) has given some values for iron and silicon and in this paper estimates for sulphur are given. Write the internal energy change as:

$$\int_V \rho \frac{\mathrm{De}}{\mathrm{Dt}}\,\mathrm{dV}=\int_V \rho T \frac{\mathrm{Ds}}{\mathrm{Dt}}\,\mathrm{dV}-\int_V p' \nabla \cdot \mathbf{u}\,\mathrm{dV}+\int_V \mathbf{i} \cdot \nabla \mu\,\mathrm{dV}.$$

The entropy integral is:

$$\int_V \rho T \frac{Ds}{Dt}\,\mathrm{dV}=\int_V \left[\rho T\left(\frac{\partial s}{\partial T}\right)_{P,C} \frac{\mathrm{DT}}{\mathrm{Dt}}-\rho T\left(\frac{\partial s}{\partial c}\right)_{P,T} \operatorname{div} \mathbf{i}\right] \mathrm{dV}.$$

An extra term in the internal energy is therefore the heat of reaction:

$$T\left(\frac{\partial s}{\partial c}\right)_{P,T}=-T\left(\frac{\partial \mu}{\partial T}\right)_{P,C}$$

by the Maxwell relation. Braginsky (1964) takes a large value for FeSi of 30 kcal mole^{-1} and for Fe+S → FeS it is 20 kcal mole^{-1}. The pressure dependence is unknown. For FeS this heat of reaction amounts to 1.6×10^7 joules kg^{-1}. If the outer core contains 4% by weight of sulphur, then in freezing 1 kg at the inner core boundary, 4×10^6 joules of latent heat are liberated. Of this heat, about 2×10^6 joules could be absorbed in dissociation of FeS.

The integral C is evaluated from $\nabla \mu$:

$$\nabla \mu=\left(\frac{\partial \mu}{\partial T}\right)_{C,P} \nabla T+\left(\frac{\partial \mu}{\partial c}\right)_{P,T} \nabla c+\left(\frac{\partial \mu}{\partial p}\right)_{C,T} \nabla p.$$

The largest contribution is probably the pressure gradient term, so that:

$$\nabla \mu \approx\left(\frac{\partial \mu}{\partial p}\right)_{C,T} \rho \mathbf{g}.$$

By the Maxwell relation:

$$\left(\frac{\partial \mu}{\partial p}\right)_{C,T}=\left(\frac{\partial V}{\partial c}\right)_{P,T}=-\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial c}\right)_{P,T}$$

and:

$$\nabla \mu = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial c}\right)_{P,T} \mathbf{g}.$$

The constant can be estimated roughly by an additive law, taking densities at standard temperature and pressure. It is about 0.75, similar to Braginsky's estimate for silicon. To estimate $\mathbf{i}$, the mass flux carried by diffusion only, we have:

$$\mathbf{i} \approx -\rho D \nabla c$$

where D is the diffusion coefficient. This has been measured for silicon and sulphur in liquid iron just above the melting point and for both it is about $10^{-8}\,\mathrm{m}^2\,\mathrm{s}^{-1}$. The value 2000 degrees higher could be one hundred times as big. The pressure effect is not known. Taking $\rho = 2\,\mathrm{Mg\,m}^{-3}$ and estimating ∇c by assuming a change in concentration from 4% to 2% across the core gives $\mathbf{i} \approx 10^{-13}\,\mathrm{kg\,m}^{-2}\,\mathrm{s}^{-1}$. The integral C is then about 10^8 watt. This energy is the heat released by the diffusing material. Its physical interpretation in Equation (15) is that the part of the gravitational energy released by the diffusion of light material is not available to generate magnetic field. Alternatively one may view the diffusion process as an extra means of dissipating heat, which competes with Ohmic diffusion for the available energy. The numerical estimate of 10^8 watt is not very large. Braginsky (1964) claims this effect to be important but it is not clear why, because his lower diffusion coefficient should give a smaller value.

Summary

The principal result of this paper is Equation (9), which shows that the gravitational energy released by rearrangement of matter in the core is completely converted to magnetic dissipation enabling a large magnetic field to be generated with a low heat flow from the core. A rough estimate of the gravitational energy available, based on seismic models, suggests that a modest magnetic field of about 50 Gauss could be maintained. Thermal convection is an inefficient way to generate magnetic field and involves too high a heat. The added complication of chemically reactive components may actually lower the total heat available and significantly reduce the efficiency of the dynamo because of heat dissipated in diffusion of material. The density jump at the inner core boundary determines the magnitude of the chemical separation effect, and efforts are being made to improve seismological estimates of it. Another seismologically observable parameter is the Brunt-Vaisäla frequency N, where

$$N^2 = -\frac{g\dfrac{d\rho}{dr}}{\rho} - \frac{\rho g^2}{\Lambda}$$

and Λ is the Lamé parameter. This is not only a measure of the difference between the temperature gradient and the adiabatic value, but also of the concentration gradient of light component in the mixture in the outer core. Better determinations of this parameter will give constraints on core compositions.

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