J Geophys (1985) 56: 100–105

Journal of Geophysics

The normal modes of a uniform, compressible Maxwell half-space

Detlef Wolf

Department of Physics, University of Toronto, Toronto, Ontario, Canada, MSS 1A7

Abstract. The analytical solution for the load-induced deformation of a uniform, compressible, hydrostatically pre-stressed elastic half-space is derived. The solution is correct to first order in the quantity ε, which is inversely proportional to the wave number k of the deformation. Usually ε is very small compared with unity for Earth deformations on a scale amenable to the half-space approximation. Since pre-stress advection is included in the analysis, the correspondence principle allows us to solve the field equations governing the deformation of the associated Maxwell half-space. The viscoelastic solution shows that the relaxation of the Maxwell continuum is characterized by a fundamental mode and a rapidly decaying overtone of much smaller amplitude. In the incompressible limit the overtone is not excited. The significance of the results for the relaxation of the Earth's mantle is briefly discussed.

Key words: Compressibility – Maxwell continuum – Normal modes

Introduction

The response of the Earth's lithosphere or mantle to applied surface loads, such as volcanic islands, sedimentary basins or glacial loads, has frequently been modelled using Maxwell continua (e.g. Walcott, 1970; Beaumont, 1978; Lambeck and Nakiboglu, 1981; Nakiboglu and Lambeck, 1982). In these models the Earth's compressibility was, however, usually neglected. This simplification was probably motivated by the view that allowing for compressibility would be unlikely to change the results markedly.

In a series of papers starting in 1974, Peltier developed a general theory for the relaxation of self-gravitating, compressible Maxwell Earth models. The formalism was applied to infer the Earth's viscosity stratification from deglaciation-induced relative-sea-level variations (see Peltier, 1982, for a summary).

One of several interesting aspects of Peltier's investigations was the recognition of the complicated response pattern of "realistic" Maxwell Earth models whose elastic structure is taken from seismological Earth models. The relaxation of realistic models is characterized by a multitude of discrete and exponentially decaying modes carrying distinct proportions of the total strain energy (Peltier, 1976; Wu and Peltier, 1982).

Each non-adiabatic density contrast, for example, is associated with a characteristic mode. This is also a feature of purely Newtonian viscous models (Parsons, 1972). Each viscosity contrast, such as the contrast near the base of the lithosphere, causes additional modes in the Maxwell model. This is a feature not parallelled in Newtonian viscous models, in which discontinuities in viscosity primarily modify the relaxation of the fundamental mantle mode associated with the density jump at the Earth's surface (McConnell, 1968). Some of the higher modes have relaxation times that are very short compared with the relaxation time of the fundamental mantle mode. Peltier (1976) therefore termed them transition modes. Usually these modes are only poorly excited, however.

Wu and Peltier (1982) also studied the effects due to compressibility on the relaxation of Maxwell continua and compared the response of compressible and incompressible Maxwell models in the Laplace-transform domain. They showed that, whereas the initial elastic response of the compressible model is characterized by significantly enhanced deformation, the final inviscid response is identical to that of the incompressible approximation.

In the following we will further examine the modifications introduced by compressibility. For this purpose we will study the relaxation of a particularly simple Maxwell Earth model. The analysis will be based on the formal solution for a uniform, compressible and pre-stressed elastic half-space. Application of the correspondence principle and normal-mode analysis will then allow us to show that the associated Maxwell half-space is characterized by the usual fundamental mode *and* an "overtone" of short relaxation time. The geophysical consequences of this will be briefly discussed by considering a characteristic numerical example.

Theory

Although the model of a uniform Maxwell half-space is elementary, it may serve as a first approximation when studying deformations of the Earth's mantle on a timescale characteristic of deglaciation events. The solution of the equivalent elastic problem is published in several textbooks (e.g. Jeffreys, 1976, pp. 265–267). If the elastic solution is to be used to solve the associated Maxwell problem, it must be modified, however, and gravitational restoring forces must be included.

This has recently been discussed for incompressible continua (Wolf, 1985a, b). In this approximation the governing equations can be re-formulated in terms of the total perturbation stress. If the continuum is compressible, this simple method fails and a more general approach is required. In the following, we will transform the field equations governing the deformation of a compressible, pre-stressed elastic half-space into a simultaneous first-order differential system. This system can be solved using standard matrix methods.

We confine ourselves to axisymmetric loading problems and use cylindrical co-ordinates r, ϕ, z. Then the stress-strain relations are

$$\sigma_{rz}=\mu\left(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right), \tag{1}$$

$$\sigma_{zz}=(\lambda+2\mu)\frac{\partial w}{\partial z}+\frac{\lambda}{r}\frac{\partial}{\partial r}(r u). \tag{2}$$

Taking the first-order Hankel transform of Eq. (1) and the zeroth-order Hankel transform of Eq. (2), with respect to the radial co-ordinate r, we obtain

$$\hat{u}'_1-k\,\hat{w}_0-\frac{1}{\mu}\hat{\sigma}_{rz1}=0, \tag{3}$$

$$\frac{\lambda k}{\lambda+2\mu}\hat{u}_1+\hat{w}'_0-\frac{1}{\lambda+2\mu}\hat{\sigma}_{zz0}=0. \tag{4}$$

Here u, w, σ_{rz} and σ_{zz} denote the radial and vertical displacement components and the appropriate *elastic* stress components, respectively. Parameters λ and μ are Lamé's first and second constants. A circumflex denotes Hankel transformation of zeroth or first order, as indicated by the subscript; symbol k denotes the Hankel-transform variable or wave number. A prime is used to indicate differentiation with respect to the vertical co-ordinate z.

The two components of the equilibrium equation are

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\partial\sigma_{rz}}{\partial z}+(\sigma_{rr}-\sigma_{\phi\phi})/r+\rho g\frac{\partial w}{\partial r}=0, \tag{5}$$

$$\frac{\partial\sigma_{rz}}{\partial r}+\frac{\partial\sigma_{zz}}{\partial z}+\sigma_{rz}/r+\rho g\frac{\partial w}{\partial z}=0. \tag{6}$$

The last term in Eqs. (5) and (6), respectively, accounts for stress advection in a hydrostatically pre-stressed elastic continuum of density ρ. The external gravity field g is assumed to act in the positive z-direction. Density changes due to the dilatation of the material have been neglected. Upon first-order Hankel transformation of Eq. (5) and zeroth-order Hankel transformation of Eq. (6) we obtain

$$-\frac{4(\lambda+\mu)\mu k^2}{\lambda+2\mu}\hat{u}_1-\rho g k\hat{w}_0+\hat{\sigma}'_{rz1}-\frac{\lambda k}{\lambda+2\mu}\hat{\sigma}_{zz0}=0, \tag{7}$$

$$\rho g\hat{w}'_0+k\hat{\sigma}_{rz1}+\hat{\sigma}'_{zz0}=0, \tag{8}$$

where σ_{rr} and $\sigma_{\phi\phi}$ have been eliminated using the appropriate stress-strain relations.

Equations (3), (4), (7) and (8) may be written in matrix form. With $D=d/dz$ we obtain

$$\begin{bmatrix} D & -k & -\frac{1}{\mu} & 0 \\ \frac{\lambda k}{\lambda+2\mu} & D & 0 & -\frac{1}{\lambda+2\mu} \\ -\frac{4(\lambda+\mu)\mu k^2}{\lambda+2\mu} & -\rho g k & D & -\frac{\lambda k}{\lambda+2\mu} \\ 0 & \rho g D & k & D \end{bmatrix}\begin{bmatrix}\hat{u}_1\\ \hat{w}_0\\ \hat{\sigma}_{rz1}\\ \hat{\sigma}_{zz0}\end{bmatrix}=0. \tag{9}$$

This is the generalization, for a pre-stressed continuum, of the first-order system derived by Farrell (1972). If we assume solutions of the type $\exp(mz)$, the attenuation constants m will be roots of the secular determinant

$$(\lambda+2\mu)m^4-2(\lambda+2\mu)k^2m^2+(\lambda+2\mu)k^4$$
$$+\rho g m^3-\rho g k^2 m=0. \tag{10}$$

We find

$$m_{1,2}=\pm k, \tag{11}$$

$$m_{3,4}=-\frac{\lambda+\mu}{2(\lambda+2\mu)}\varepsilon k\pm k\left[1+\frac{(\lambda+\mu)^2\varepsilon^2}{4(\lambda+2\mu)^2}\right]^{1/2}, \tag{12}$$

where $\varepsilon=\rho g/[(\lambda+\mu)k]$ has been introduced. As discussed by Cathles (1975, pp. 38–39), ε will in general be very small compared with unity for deformations of the Earth whose scale is sufficiently small to be modelled by half-space approximations. Neglecting higher-order terms, we therefore have

$$m_{3,4}=\pm k\left[1\mp\frac{\lambda+\mu}{2(\lambda+2\mu)}\varepsilon\right], \tag{13}$$

which is correct to first order in ε.

If pre-stress is neglected, $g=0$ and therefore $\varepsilon=0$. If, on the other hand, the continuum is incompressible, $\lambda\to\infty$ and again $\varepsilon=0$. In both cases the secular determinant, Eq. (10), has two double roots, and Eq. (9) represents a degenerate system. The *combined* effects of compressibility and pre-stress advection therefore remove the degeneracy of the system and cause a "gravitational splitting" of the attenuation constants, which has some similarity with the rotational splitting of the eigenfrequencies in the theory of the Earth's free oscillations. From Eq. (13) it is also evident that the attenuation of the elastic field quantities with depth is not solely determined by the lateral scale of the load but is also influenced by the material properties of the elastic continuum.

The eigenfunctions belonging to the four eigenvalues given by Eqs. (11) and (13) are calculated using matrix methods described by Frazer et al. (1938, pp. 61–70, 156–172). We obtain, correct to first order in ε,

$$\begin{bmatrix}\hat{u}_1\\ \hat{w}_0\\ \hat{\sigma}_{rz1}\\ \hat{\sigma}_{zz0}\end{bmatrix}=\begin{bmatrix}\mp 1-\varepsilon\\ 1\\ -2\mu k\mp\mu k\varepsilon\\ \pm 2\mu k-\lambda k\varepsilon\end{bmatrix}A_{1,2}\exp(m_{1,2}z), \tag{14}$$

$$\begin{bmatrix} \hat{u}_1 \\ \hat{w}_0 \\ \hat{\sigma}_{rz1} \\ \hat{\sigma}_{zz0} \end{bmatrix} = \begin{bmatrix} \mp 1 - \varepsilon/2 \\ 1 \pm \dfrac{\mu}{2(\lambda+2\mu)}\varepsilon \\ -2\mu k \mp \mu k \dfrac{\mu}{\lambda+2\mu}\varepsilon \\ \pm 2\mu k - \lambda k \varepsilon \end{bmatrix} A_{3,4} \exp(m_{3,4} z). \tag{15}$$

The following analysis will be limited to a uniform half-space. To be consistent with the direction of the gravity field adopted in Eqs. (5) and (6), the continuum must occupy the region $z>0$. If we impose the usual boundary conditions,

$$\hat{\sigma}_{rz1}(z=0)=0, \tag{16a}$$

$$\hat{\sigma}_{zz0}(z=0)=-\hat{q}_0, \tag{16b}$$

we obtain, from Eqs. (14) and (15),

$$A_2=\frac{2(\lambda+2\mu)-\mu\varepsilon}{(\lambda+\mu)\varepsilon}\,\frac{\hat{q}_0}{2\mu k+\lambda k\varepsilon}, \tag{17}$$

$$A_4=\frac{-2(\lambda+\mu)+(\lambda+2\mu)\varepsilon}{(\lambda+\mu)\varepsilon}\,\frac{\hat{q}_0}{2\mu k+\lambda k\varepsilon}. \tag{18}$$

The solution for the vertical surface displacement, the quantity of prime geophysical interest, then becomes

$$\hat{w}_0(z=0)=\hat{q}_0\left[1+(2-\varepsilon)\frac{\mu}{2(\lambda+\mu)}\right]\Big/(2\mu k+\lambda k\varepsilon). \tag{19}$$

If pre-stress is neglected, $\varepsilon=0$ and

$$\hat{w}_0(z=0)=(\lambda+2\mu)\,\hat{q}_0/[2\mu k(\lambda+\mu)], \tag{20}$$

which is the familiar solution for the non-gravitating half-space. If compressibility is neglected, $\lambda\to\infty$ and

$$\hat{w}_0(z=0)=\hat{q}_0/(2\mu k+\rho g). \tag{21}$$

This is identical to the solution discussed in Wolf (1985b), which was, however, derived directly from the incompressible field equations. Equation (21) is correct without restrictions on the wave number k. Since we will be applying our elastic solution to solve the corresponding Maxwell problem, it is illuminating to reduce Eq. (19) to its inviscid limit, $\mu=0$, which is also the infinite-time limit for Maxwell continua (Wu and Peltier, 1982). Then

$$\hat{w}_0(z=0)=\hat{q}_0/(\rho g), \tag{22}$$

which expresses local compensation of the load by buoyancy forces.

If $\varepsilon\ll 1$, Eq. (19) is simplified to

$$\hat{w}_0(z=0)=(\lambda+2\mu)\,\hat{q}_0/[2\mu k(\lambda+\mu)+\rho g\lambda]. \tag{23a}$$

This may also be written in terms of Poisson's ratio ν. Since $2\nu=\lambda/(\lambda+\mu)$, we obtain

$$\hat{w}_0(z=0)=\hat{q}_0(1-\nu)/(\mu k+\nu\rho g). \tag{23b}$$

This equation is slightly different from that proposed by Nakiboglu and Lambeck [1982, Eq. (24)] as the solution to the same problem which is considered here. Their solution was, however, derived from physically unreasonable boundary conditions (see Wolf, 1985) and assumed to be universally valid for any value of k.

According to the correspondence principle (Appendix A), Eq. (23a) can be interpreted as the Laplace transform $\tilde{\hat{w}}(s)$ of the impulse response of the associated Maxwell continuum. If λ and μ are replaced by Eqs. (30) and (31) and if the Laplace transform $\tilde{T}^{(ve)}(s)=\tilde{\hat{w}}(s)/\hat{q}$ of the viscoelastic transfer function is introduced, Eq. (23a) is thus transformed to

$$\tilde{T}^{(ve)}(s)=\frac{3(\lambda+2\mu)s^2+2(3\lambda+4\mu)\tau^{-1}s+(3\lambda+2\mu)\tau^{-2}}{3[2\mu k(\lambda+\mu)+\rho g\lambda]s^2+2[\mu k(3\lambda+2\mu)+\rho g(3\lambda+\mu)]\tau^{-1}s+\rho g(3\lambda+2\mu)\tau^{-2}}, \tag{24}$$

where the subscript has been dropped. Taking the inverse Laplace transform (Appendix B) yields

$$T^{(ve)}(t)=T^{(e)}\delta(t)+T^{(v,1)}s^{(1)}\exp(-s^{(1)}t)+T^{(v,2)}s^{(2)}\exp(-s^{(2)}t), \tag{25}$$

for the impulsive forcing $\hat{q}(k)\,\delta(t)$ or

$$T^{(ve)}(t)=T^{(e)}-T^{(v,1)}[\exp(-s^{(1)}t)-1]-T^{(v,2)}[\exp(-s^{(2)}t)-1], \tag{26}$$

for the Heaviside loading event $\hat{q}(k)H(t)$.

The explicit formulae for the viscous transfer functions (normal modes) $T^{(v,1)}$, $T^{(v,2)}$ and the associated inverse relaxation times $s^{(1)}$, $s^{(2)}$ are not very illuminating. In the incompressible limit, $\lambda\to\infty$, however, simple analytical expressions result for the inverse relaxation times, viz.

$$s^{(1)}=\tau^{-1}, \tag{27}$$

$$s^{(2)}=\rho g\tau^{-1}/(2\mu k+\rho g). \tag{28}$$

Numerical example

The portion of the relaxation carried by the fundamental mode in the uniform model was analysed previously to some extent (Nakiboglu and Lambeck, 1982; Wu and Peltier, 1982; Wolf, 1984). The existence of an overtone with a relaxation time close to the Maxwell time [and close to the relaxation times of the transition modes of realistic Earth models identified by Peltier (1976)] in this simple model was, however, appreciated previously and is therefore also discussed here.

The effects due to compressibility are displayed in Fig. 1, where $T^{(e)}$, $T^{(v,1)}$, $T^{(v,2)}$, $s^{(1)}$ and $s^{(2)}$ are plotted as functions of wave number. For definiteness we have taken $\lambda=0.80\times10^{11}\,\mathrm{N\,m^{-2}}$, $\mu=0.67\times10^{11}\,\mathrm{N\,m^{-2}}$ and $\rho=3{,}380\,\mathrm{kg\,m^{-3}}$. These values are fairly characteristic of the Earth at 100-km depth (Bullen, 1963, pp. 232–235). The viscosity is $\eta=10^{21}$ Pa s, which appears to be typical of the upper mantle (e.g. Cathles, 1975). A cut-off angular order of $n=ka=5$ has been chosen for the

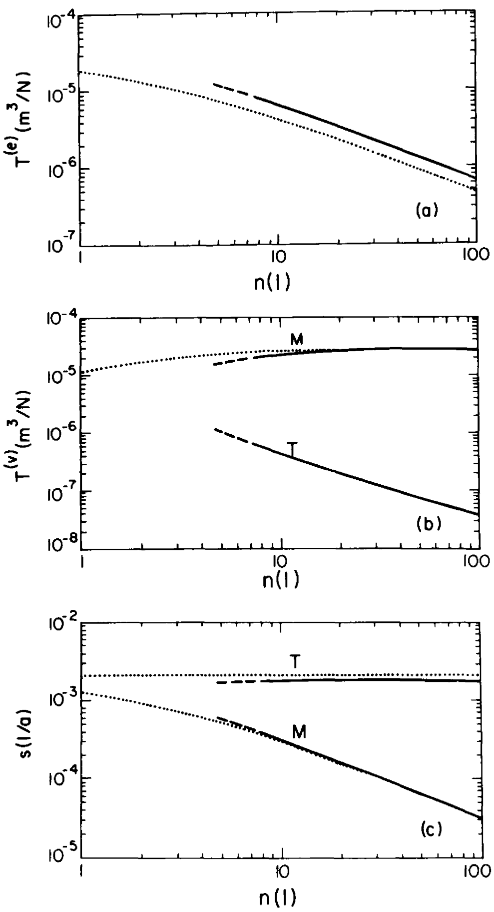

Fig. 1a–c. a Elastic transfer function $T^{(e)}$, **b** viscous transfer function $T^{(v)}$ and **c** inverse relaxation time s as function of angular order n according to compressible solution (*solid*) or incompressible approximation (*dotted*); symbols M and T denote relaxation mode

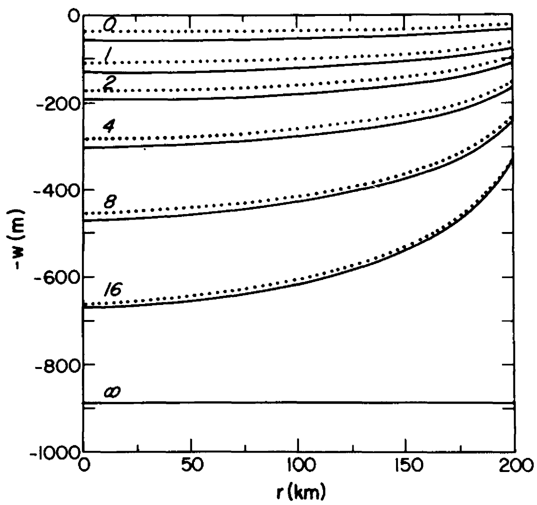

Fig. 2. Vertical surface displacement w as function of distance r from load axis for several times (in units of ka) after emplacement of load according to compressible solution (*solid*) and incompressible approximation (*dotted*)

compressible model, where a is the Earth's radius. For this value of n we find $\varepsilon = 0.29$ and thus $\varepsilon^2 = 0.08$.

The elastic transfer function $T^{(e)}$ is illustrated in Fig. 1a. Inspection of the diagram shows that $\nu = 0.272$ (which corresponds to $\lambda = 0.80 \times 10^{11}\,\mathrm{N\,m^{-2}}$) leads to an increase in $T^{(e)}$ by approximately 40% compared with $\nu = 0.5$ (which corresponds to $\lambda \to \infty$). This is what is required by Eq. (23b).

The relaxation of a particular deformation is governed by the spectral characteristics of the viscous response. Figure 1c shows that the neglect of compressibility causes an insignificant change in the relaxation times of either mode. This has already been noted for the fundamental mode (Wu and Peltier, 1982).

In Fig. 1b the amplitude spectra $T^{(v,1)}$ and $T^{(v,2)}$ are displayed. The fundamental mode of the compressible model is characterized by reduced amplitude compared with the incompressible approximation. Since the sum $T^{(e)} + T^{(v,1)} + T^{(v,2)}$ is independent of the value adopted for λ [see Eq. (22)], this is consistent with the fact that the compressible model's elastic response is enhanced (Fig. 1a).

The overtone is only excited if $\lambda < \infty$. It is therefore intimately related to the compressibility of the material. For $\lambda = 0.80 \times 10^{11}\,\mathrm{Nm^{-2}}$, it carries a small fraction of the total viscous response. The relative strength of the overtone increases with decreasing angular order. This trend is expected to continue for $n < 5$, which is, however, below the range of validity of our approximation.

In order to assess the geophysical significance of these differences, we study the response in the spatial domain for a square-edged disk load of 200-km radius, 3-km thickness and a density of $1{,}000\,\mathrm{kg\,m^{-3}}$, and for the Heaviside loading event $\hat{q}(k)\,H(t)$. In taking the inverse Hankel transform (Appendix C), the limits of integration have been fixed to $n = 5$ and $n = 1{,}000$. The truncation at the lower end primarily affects the accuracy of the response near the elastic limit. The behaviour at large times is more influenced by the cut-off at $n = 1{,}000$. The total truncation error is, however, small and amounts to a few per cent at the most.

In Fig. 2 the vertical surface deflection is shown. Since the distribution of the model parameters is uniform, a peripheral bulge does not develop as relaxation proceeds (Nakiboglu and Lambeck, 1982; Wolf, 1984). The discussion has therefore been confined to the central region below the load. From inspection of the figure it is evident that effects due to compressibility are noteworthy only during the initial period of relaxation following the emplacement of the load. At $t = 4$ ka the differences in deflection have already decreased to approximately 10%; at $t = 16$ ka the discrepancy is only a few per cent.

Conclusion

The geophysical significance of compressibility is clearly contingent upon whether the earlier or later phases of relaxation are sampled by the observations. In interpretations of glacio-isostatic rebound, for example, the lithosphere is usually regarded as elastic and the adjustments are mainly controlled by the viscosity of the Earth's mantle. Since relaxation times are characteristically between 1 and 10 ka (Wu and Peltier, 1982, Fig. 10) and since deglaciation was almost complete at 8 ka before present, the majority of the post-glacial-adjustment data sample the intermediate or later phases of relaxation. During the early phases when effects due to compressibility are more noticeable, relaxation is also markedly influenced by the unloading event itself. The shape of the ice-sheet and the details of the deglaciation history are, however, not known with great accuracy. Consequently, the interpretation of isostatic-adjustment data from this time interval is subject to considerable uncertainties (e.g. Wolf, 1985c). We may therefore conclude that incompressible Maxwell continua are adequate representations of the Earth's mantle in most instances.

Appendix A

Correspondence principle

According to the correspondence principle (Biot, 1954; Peltier, 1974; Cathles, 1975, pp. 23–29), the solution to an elasto-static problem can be interpreted as the Laplace transform of the quasi-static response, to an impulsive load $q\,\delta(t)$, of the associated Maxwell continuum governed by Laplace-transformed stress-strain relations

$$\tilde{\sigma}_{ij}=\lambda(s)\,\tilde{\sigma}_{kk}\,\delta_{ij}+2\,\mu(s)\,\tilde{\varepsilon}_{ij}, \tag{29}$$

provided that

$$\mu(s)=\lambda s/(s+\tau^{-1}), \tag{30}$$

$$\lambda(s)=(\lambda s+K\tau^{-1})/(s+\tau^{-1}). \tag{31}$$

Here the tilde denotes Laplace transformation with respect to time t; s is the Laplace-transform variable. $K=\lambda+2\mu/3$ denotes the bulk modulus and $\tau=\eta/\mu$ the Maxwell time, with η the dynamic viscosity. Since large values of s correspond to short-time-scale behaviour and vice versa, we realize, from inspection of Eqs. (29)–(31), that the instantaneous response, $t=0$, is elastic. Constitutive relations appropriate to long times after the loading event are obtained by observing that $K(s)=\lambda(s)+2\,\mu(s)/3=K$, which is readily verified from Eqs. (30) and (31). Then Eq. (29) may be written in the alternative form

$$\tilde{\sigma}_{ij}=K\,\tilde{\varepsilon}_{kk}\,\delta_{ij}-2/3\,\mu(s)\,\tilde{\varepsilon}_{kk}\,\delta_{ij}+2\,\mu(s)\,\tilde{\varepsilon}_{ij}. \tag{32}$$

In the limit of $t\to\infty$ Eq. (30) vanishes, and Eq. (32) takes the form

$$\tilde{\sigma}_{ij}=K\,\tilde{\varepsilon}_{kk}\,\delta_{ij} \tag{33a}$$

or, in the time domain,

$$\sigma_{ij}=K\,\varepsilon_{kk}\,\delta_{ij}. \tag{33b}$$

These are the inviscid constitutive relations for a compressible continuum.

Appendix B

Normal modes

The Laplace transform $T^{(ve)}(s)$, of the viscoelastic transfer function describing the relaxation of the Maxwell continuum following an impulsive forcing $\hat{q}\,\delta(t)$, can be split into an elastic portion $T^{(e)}$ and a viscous portion $\tilde{V}(s)$. We may therefore write

$$\tilde{T}^{(ve)}(s)=T^{(e)}+\tilde{V}(s), \tag{34}$$

where

$$T^{(e)}=\lim_{s\to\infty}\tilde{T}^{(ve)}(s). \tag{35}$$

Here $T^{(e)}$ represents the Laplace transform of the instantaneous elastic response of the viscoelastic continuum to the impulsive load. The transform $\tilde{V}(s)$ of the viscous portion can be cast into the form

$$\tilde{V}(s)=\frac{T^{(v,1)}s^{(1)}}{s+s^{(1)}}+\frac{T^{(v,2)}s^{(2)}}{s+s^{(2)}}, \tag{36}$$

where $T^{(v,1)}$, $T^{(v,2)}$ and $s^{(1)}$, $s^{(2)}$ are complicated functions of the model parameters. Taking the inverse Laplace transform yields

$$T^{(ve)}(t)=T^{(e)}\,\delta(t)+T^{(v,1)}s^{(1)}\exp(-s^{(1)}t)+T^{(v,2)}s^{(2)}\exp(-s^{(2)}t). \tag{37}$$

This is the system's impulse response. The response to a Heaviside loading event $\hat{q}\,H(t)$ follows from convolving it with the impulse response. From Eq. (37) we obtain, for $t\geqq 0$,

$$T^{(ve)}(t)=T^{(e)}-T^{(v,1)}[\exp(-s^{(1)}t)-1]-T^{(v,2)}[\exp(-s^{(2)}t)-1]. \tag{38}$$

Appendix C

Inverse Hankel transform

We are concerned with square-edged, circular disk loads

$$q(r)=\begin{cases}0, & 0\leqq r<R\\ 1, & R<r<\infty\end{cases}, \tag{39}$$

where R is the radius of the disk. The load distribution $q(r)$ can be written as the inverse zeroth-order Hankel transform

$$q(r)=\int_0^\infty \hat{q}(k)\,k\,J_0(kr)\,dk, \tag{40}$$

where (e.g. Sneddon, 1951, p. 528)

$$\hat{q}(k)=J_1(kR)\,R/k. \tag{41}$$

The response in the spatial domain is then obtained from

$$w(r,0)=\int_0^\infty T^{(ve)}(k)\,\hat{q}(k)\,k\,J_0(kr)\,dk, \tag{42}$$

for the half-space, where $T^{(ve)}(k)$ denotes the viscoelastic transfer function.

Acknowledgements. This research was financially supported by a Natural Sciences and Engineering Research Council of Canada Postgraduate Scholarship. Discussions with Richard Peltier have been very helpful.

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Received June 5, 1984; Revised version November 5, 1984
Accepted November 6, 1984