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Shear-Wave Singularities of Wave Propagation in Anisotropic Media

S. Crampin [1], and M. Yedlin [2]

[1] Institute of Geological Sciences, Murchison House, West Mains Road, Edinburgh EH9–3LA, Scotland
[2] Department of Physics, University of Alberta, Edmonton, Canada

Abstract. Shear-wave singularities in systems of anisotropic symmetry are comparatively well known, but it has not been generally realised that they may cause anomalies in shear-wave propagation for neighbouring directions due to the behaviour of the polarizations. Singularities are places where the two shear-wave slowness-surfaces are continuous with each other through common points. The most frequent type, a point singularity, is a place where the two surfaces are continuous with each other through the vertices of cone-shaped projections from the surfaces. For directions of propagation in a plane, which cuts the slowness surfaces near a singularity, the velocities of the two shear-waves approach each other in a pinch and at the pinch exchange polarizations and velocity gradients. These singularities do not cause anomalies in plane waves propagating in a uniform medium, but may cause mode conversion and pulse-shape modification to waves with spherical wave-fronts, and to rays of shear-waves, in varying anisotropic media.

Key words: Shear-wave singularities – Point-, kiss-, and intersection-singularities – Pinches – Anisotropic shear-waves – Shear-wave polarizations.

Introduction

There are three orthogonally-polarized body-waves propagating in every direction in anisotropic elastic solids (Crampin 1977); a quasi *P*-wave, *qP*, and two quasi shear-waves, *qS1* and *qS2*, or *qSH* and *qSV* if appropriate. The velocities of these waves vary with the direction, and are roots of an equation, which can either be written as a third-order polynomial in V^2 (Synge 1957), or a third-order eigen-value problem in V^2 (Crampin 1977).

The three body-waves in each direction are generally distinct and their velocities vary independently with direction. However, the two shear-wave slowness-surfaces are analytically continuous, which, in some directions, may introduce complications into the shear-wave propagation. The easiest way to observe this continuity is to trace the intersections of the shear-wave slowness-sheets with the three orthogonal symmetry-planes round an orthogonal corner of an orthorhombic material. It can be demonstrated that polarizations of the two shear waves, propagating parallel to a symmetry plane, are parallel and perpendicular, respectively, to the symmetry plane (Crampin and Kirkwood 1980, henceforth called Paper 1). Hence, following the polarizations of one of the slowness sheets, the inner sheet say, round the orthogonal corner, the symmetry-plane polarizations indicate that the intersections must cross each other an odd number of times. The shear waves of orthorhombic orthopyroxene (Fig. 6, Paper 1), for example, cross three times, so that on completion of the corner we are now on the outer sheet. The places where the waves cross are singularities of the shear-wave surfaces.

Singularities in shear-wave slowness-sheets have been recognised for many years (Duff 1960). Indeed, singularities of various kinds are very common for shear waves in anisotropic media. Symmetry considerations indicate that the two shear-wave slowness-sheets of orthopyroxene have 12 singularities as just described, and the two sheets of a cubic solid, which one might have thought was a comparatively simple structure, have 14 singularities of various kinds (Paper 1). What has not been recognised, however, is the behaviour of shear-wave polarizations near such singularities and the disturbance this may cause to propagating shear-waves in neighbouring directions. This paper investigates the properties of singularities and the behaviour of shear waves at the various types of singularity. It is meant to be read in parallel with Paper 1, which lists the singularities and other properties of the various systems of anisotropic symmetry.

It is appropriate to publish this work in the geophysical literature, because anisotropy may be important for mapping structures in the crust (Crampin et al. 1980b). In particular, investigating shear-wave polarization-anomalies appears to be an important technique for estimating upper-mantle anisotropy (Crampin 1977; Ando et al. 1980), and for monitoring the progress of earthquake dilatancy (Crampin 1978; Crampin et al. 1980a).

Types of Singularity and Their Effect on Shear-Wave Polarization

Duff (1960) demonstrated that the inner, *qP*, slowness sheet is convex and wholely interior to the shear-wave slowness-sheets. The two shear-wave sheets, however, come into contact at least twice in anisotropic symmetry systems, and usually much more frequently (Paper 1). The directions where the sheets are in contact are singularities of the shear-wave propagation. In these directions, the shear waves have coincident phase-velocities, possible group-velocities form a cone about the phase-velocity direction, and polarizations are any two vectors mapping out the plane perpendicular to the *qP* polarization.

There are three main types of singularity. The first two types, point- and kiss-singularities, mark discontinuities in the shear-wave polarizations, where the polarizations on both sheets vary rapidly with direction about the singularity. These singularities

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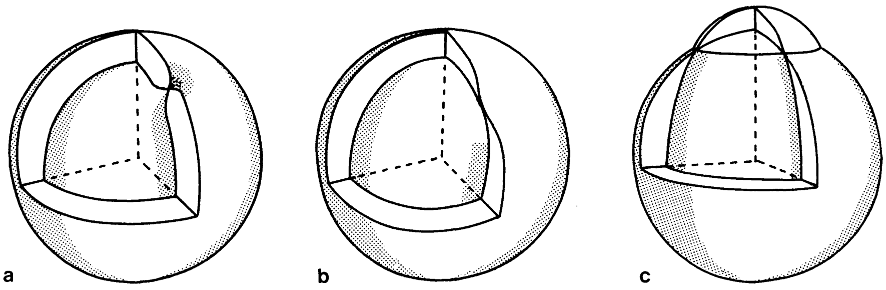

Fig. 1 a–c. Diagrams of the two shear-wave slowness-surfaces showing the topology near the various types of singularity: **a** Point singularity, **b** kiss singularity (class i), and **c** intersection singularity

may cause complications to shear-wave propagation in neighbouring directions. The third type, an intersection singularity, is a formal singularity only, and causes no particular complication in shear-wave propagation.

Point Singularities

The velocity variations in the symmetry planes of the various anisotropic symmetry systems frequently show the two shear-waves crossing each other (Paper 1). In hexagonal symmetry systems, the crossing marks the simple intersection of the two shear-wave slowness-sheets (see below). In all other symmetry systems, the crossing marks a *point singularity*, where the two slowness-sheets are continuous through the vertices of cone-shaped projections from the surfaces. The topological configuration is shown in Fig. 1a.

Point singularities are observed to lie in at least one symmetry plane. The velocity variations of each of the two orthogonally-polarized shear-waves in such symmetry planes behave no differently in the direction of the singularity from other directions in the plane. In particular, the polarizations of the two shear-waves, parallel and perpendicular, respectively, to the symmetry plane, appear continuous across the singularity. However, in neighbouring planes, which-pass near but avoid the singularity, the velocity variations of the shear waves approach each other in a *pinch*, and at the pinch, the waves on the inner and outer slowness-sheets exchange polarizations, and velocity gradients. Figure 2a shows the behaviour of the shear-wave velocities and polarizations in one of the diagonal symmetry-planes of cubic silicon (see also Fig. 2, Paper 1), and in some neighbouring planes. The pinches may be extremely sharp for variations in planes passing very near to point singularities, and the exchange of polarizations take place within a very narrow range of directions.

Similar exchanges occur between modes in Generalized surface-wave propagation in anisotropic layered media. The polarizations and gradients of the velocity variations may be exchanged as the dispersion curves of the two modes approach each other in a pinch (Crampin and Taylor 1971). This happens in layered anisotropic structures, where the dispersion curves of Rayleigh and Love modes cross each other in equivalent isotropic structures.

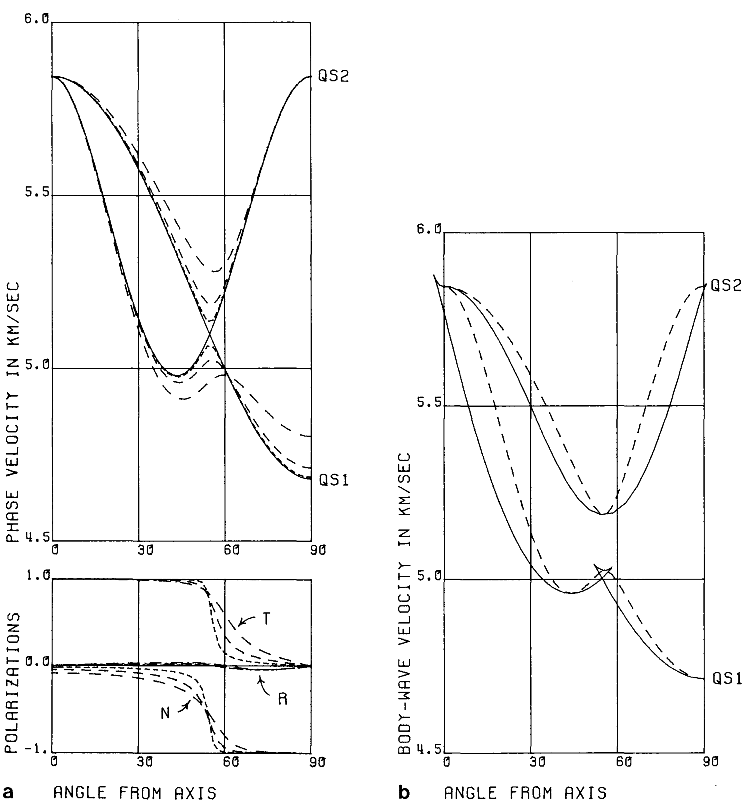

Fig. 2a and b. Variations in cubic silicon near a point singularity in a diagonal symmetry plane (see Paper 1). **a** *Upper figure*: Phase-velocity variations: *solid line* – diagonal symmetry plane, showing a point singularity; *dotted line* – plane 2° off symmetry-plane, showing tight pinch; *dashed line* – plane 5° off symmetry-plane, showing pinch; and *broad dash* – plane 10° off symmetry-plane. *Lower figure:* Direction cosines of the polarization of the slower shear-wave in the off-symmetry planes, relative to normal (N), radial (R), and transverse (T) directions to the plane of the phase-velocity variations.
b Phase and group velocity in the plane 5° off symmetry-plane: *dashed line* – phase velocity, as dashed line in (**a**) above; and *solid line* – group velocity showing cups.
Note: the *solid line* is the projection of the non-planar group-velocity on the plane of the phase-velocity variation

The physical effects of a point singularity on wave propagation will depend on the behaviour of the group or wave velocity, rather than the phase velocity. Since both the gradient and the velocity are exchanged at a pinch, the group velocity will also be exchanged. Pinches will have no effect on plane waves propagating in uniform structures, but may have large effects on spherical wavefronts and rays of shear waves through varying anisotropic structures. As the direction of propagation of a ray sweeps through a pinch, energy will tend to jump to the body wave which continues with the same polarization, and there will be an increase or decrease of velocity.

The exact effects of this phenomenon will very much depend on the particular configuration of the pinch and the speed that the ray sweeps the pinch. In some configurations, the behaviour may be similar to that illustrated in Fig. 14b of Crampin (1978), where the energy of a shear wave with one polarization is largely transferred (by mode conversion), to the shear wave with the orthogonal polarization, on propagating through a homogeneous anisotropic medium. In other circumstances, the pulse shape of shear waves will be broadened as continuous constructive and destructive interference takes place. This type of behaviour at pinches may lead to modification and attenuation of the primary shear-wave pulse.

The rapid variation of the phase velocity near a pinch may introduce cusps into the group-velocity surface (usually called the wave surface). Figure 2b shows a section of the wave surface for one of the phase-velocity variations in Fig. 2a. Strictly, the group velocity associated with the planar phase-velocity variation is not co-planar, but the deviations are usually small, as in this case, and we show the projection on the phase-velocity plane. In Fig. 2b there are cusps on the slower shear-wave sheet (*qS1*) at the pinch and at the axis, and on the faster sheet (*qS2*) at 90° from the axis. The behaviour at cusps, caused both by overall velocity variations and by variations at pinches, produces rapid variations in the ray. The direction of a ray through a varying structure depends on two phenomena. The behaviour of the phase velocity is determined by applying Snell's Law to the phase-velocity directions, and the relationship of the ray direction to the phase direction is determined by the anisotropic structure of the media. Since a cusp is a place where there are rapid changes in group-velocity direction for slower variations in phase-velocity, a cusp can produce rapid changes in the direction of a ray.

It is interesting to note that Fig. 2b provides an example of the sensitivity of cusps to the curvature of the phase-velocity sheets (strictly, the curvature at inflexions in the phase-slowness sheets). In order of increasing curvature, there is no cusp on the *qS2* sheet at the axis (0°), a small cusp at 90°, and a distinct cusp on the *qS1* sheet at 0°.

Kiss Singularities

A *kiss singularity* is a point where the two shear-wave slowness-surfaces touch tangentially, with any combination of convex or concave curvature, but do not intersect. Three classes of kiss can be distinguished:

(i) Figure 1b illustrates the simplest topological configuration of a kiss: the place where the two slowness-sheets of a hexagonal medium touch in the direction of the symmetry axis (Paper 1). The polarizations of the shear waves on these two slowness-sheets are parallel and perpendicular, respectively, to planes through the symmetry axis. The polarizations have cylindrical symmetry about the symmetry axis.

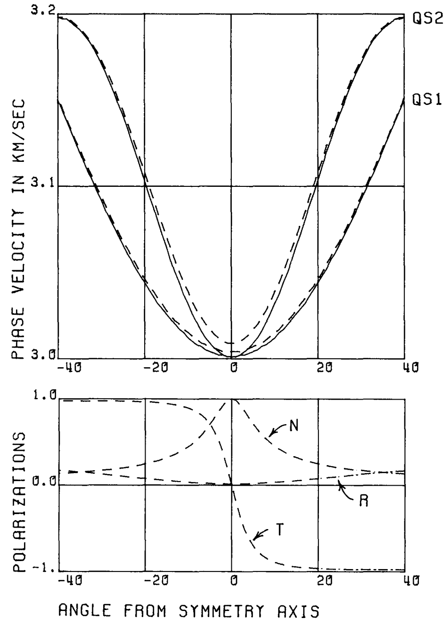

Fig. 3. Variations in hexagonal GKFF1 (Paper 1) near a kiss singularity. *Upper figure:* Phase-velocity variations. *Solid line* – variations in a plane through the symmetry axis showing a kiss singularity; and *dashed line* – variations in a plane 5° off symmetry-axis, showing a broad pinch. *Lower figure:* Direction cosines of the polarization of the faster shear-wave in the off-symmetry plane. Notation as in Fig. 2a

(ii) Kiss singularities may also occur in directions parallel to the axis of intersection of several symmetry-planes, as in cubic and tetragonal symmetry systems (Paper 1). Figure 2a shows the behaviour at this class of kiss, where the shear waves meet the axis. The polarizations do not have cylindrical symmetry, but do have reflection symmetry in the axis.

(iii) Kiss singularities may also occur in non-symmetry directions in symmetry planes, as at about 55° from the *z*-axis in *y*-cut orthorhombic orthopyroxene (Fig. 6b, Paper 1). Note that kisses of this class are not, in any sense, centres of symmetry.

Similarly to the behaviour of velocity variations near point singularities, velocity variations in planes which pass near, but not through, the first two types of kiss singularity (i and ii, above) approach each other in a broad pinch. Near this pinch the polarizations show rapid but temporary deviations. Figure 3 shows the velocities and polarizations near a kiss singularity in the hexagonal medium GKFF1 (Paper 1). Both these types of kiss singularity may cause disturbances to shear-wave propagation, as discussed in the preceding section. The third type of kiss (iii) is believed to cause no disturbance to wave propagation.

Intersection Singularities

Intersection singularities are the intersection of two shear-wave slowness-surfaces in the usual sense of two surfaces intersecting. Such singularities only occur in hexagonal symmetry-systems, where the two sheets may intersect in a circle about the symmetry axis. Figure 1c shows the topology of such an intersection. The polarizations of the shear waves on the two sheets are parallel and perpendicular to planes through the symmetry axis. There are no disturbances to wave propagation.

The two shear-wave slowness sheets, in anisotropic systems with symmetry very close to hexagonal, may separate at the position of the hexagonal intersection, and the two sheets pinch together in a nearly circular ring. The polarizations and velocity gradients of the two shear-wave sheets are interchanged at the *ring pinch,* and the phenomenon may cause disturbances to shear-wave propagation.

Discussion

Shear-wave singularities are very common in anisotropic symmetry-systems (Paper 1). Plane waves will not be disturbed by the proximity of a singularity even when propagating directly at the singularity. However, spherical wavefronts or rays of shear waves may be considerably disturbed, when the direction of propagation passes near to a point singularity or near to the first two classes of kiss singularity. Depending on the particular wave-path geometry, the wave form may be modified and attenuated by mode conversion, and constructive and destructive interference caused by cusps in the group-velocity variations and by rapid changes in the fixed polarizations.

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