J Geophys (1987) 61: 12–20

Journal of Geophysics

Analytical model calculations for heat exchange in a confined aquifer

R. Schulz

Niedersächsisches Landesamt für Bodenforschung, Postfach 51 01 53, D-3000 Hannover 51, Federal Republic of Germany

Abstract. Model calculations are carried out for the temperature distribution in an aquifer, into which water is injected. The models are simplified so that the temperature distribution can be given by an analytical expression. The one-dimensional simulation shows the well-known fact that effects of conduction can generally be neglected. In two-dimensional models the case of a doublet, in particular, is analysed. A new approach is used for solving the heat transport equation. The distributions of temperature, streamlines and potential, even if a natural uniform flow of groundwater exists, can be easily calculated and plotted. The computation of the temperature in the extraction well shows that the economical working time of a doublet can be longer than the thermal breakthrough time.

Key words: Geothermal energy – Heat transport equation – Groundwater – Hydrothermal heat transport

Introduction

The geothermal resources and reserves in the Federal Republic of Germany have been estimated for selected aquifers in the Northern German Basin, the Molasse Basin and the Upper Rhinegraben. This study indicates that the highest reserves are in the Molasse Basin of southern Germany (Haenel, 1985). In a current research project the geothermal reserves will be assessed more accurately for the Malm aquifer (Upper Jurassic) in the Molasse Basin. For that purpose and for other planned projects, computer modelling is a helpful tool.

The models treated in this paper are simplified so that the hydraulic and the heat transport equation can be solved by analytical methods. The one-dimensional models can be used for water injection or extraction into an aquifer; similar, but more complicated, models for solving this problem can be found in Mehlhorn (1982). The thermal effects in an aquifer caused by injection and production wells will be analysed by a two-dimensional model. The approach of Gringarten and Sauty (1975) for this model will be improved.

The analytical models are very useful for obtaining a general view of principal effects. Expenditure of computer programming and running time is small compared to numerical models. These simplified models are appropriate for carrying out model investigations for small selected areas.

Such a local model was established for two wells in the area of Saulgau (Baden-Württemberg, southern Germany), which will be published by the European Community; see also Table 1(b). The distance between the two wells in Saulgau is only 400 m. For greater distances or for regional investigations, the hydraulic and geological situation often cannot be modelled in that simplified way which is needed for an analytical approach. Then numerical modelling has to be used (e.g. see Lütkestratkötter, 1977). The numerical methods can be tested by the analytical models described here; this is intended for a further paper.

Basic equations

Fluid and heat transport in porous media are described by the three equations of momentum, mass and energy conservation. The comprehensive derivations of the basic equations are given by Bear (1972, 1979), Carslaw and Jaeger (1959) and Myers (1971).

A 1) First we assume that Darcy's law is valid

$$\mathbf{q} = -\mathbf{K} \operatorname{grad} \phi \tag{1a}$$

where

$\mathbf{q}$ $(m^3\,s^{-1}\,m^{-2})$ specific volume flux,
$\mathbf{K}$ $(m\,s^{-1})$ hydraulic conductivity (a tensor),
ϕ (m) hydraulic potential.

Since the dimension of the specific flux is a velocity, $\mathbf{q}$ is also designated as Darcy velocity $\mathbf{v}_F$. For isotropic media Eq. (1a) can be rewritten as

$$\mathbf{v}_F = -k_f \operatorname{grad} \phi \tag{1b}$$

where

k_f $(m\,s^{-1})$ value of hydraulic conductivity (a scalar).

The equation of mass conservation for the fluid in a porous medium is given by

$$\frac{\partial(n\rho_F)}{\partial t} + \operatorname{div}(\rho_F \mathbf{q}) + \rho_F \tilde{Q} = 0 \tag{2a}$$

where

n $(-)$ porosity,
ρ_F $(kg\,m^{-3})$ fluid density,
$\tilde{Q}$ $(m^3\,s^{-1}\,m^{-3})$ source rate per volume.

For the sake of simplicity the following assumptions are made:

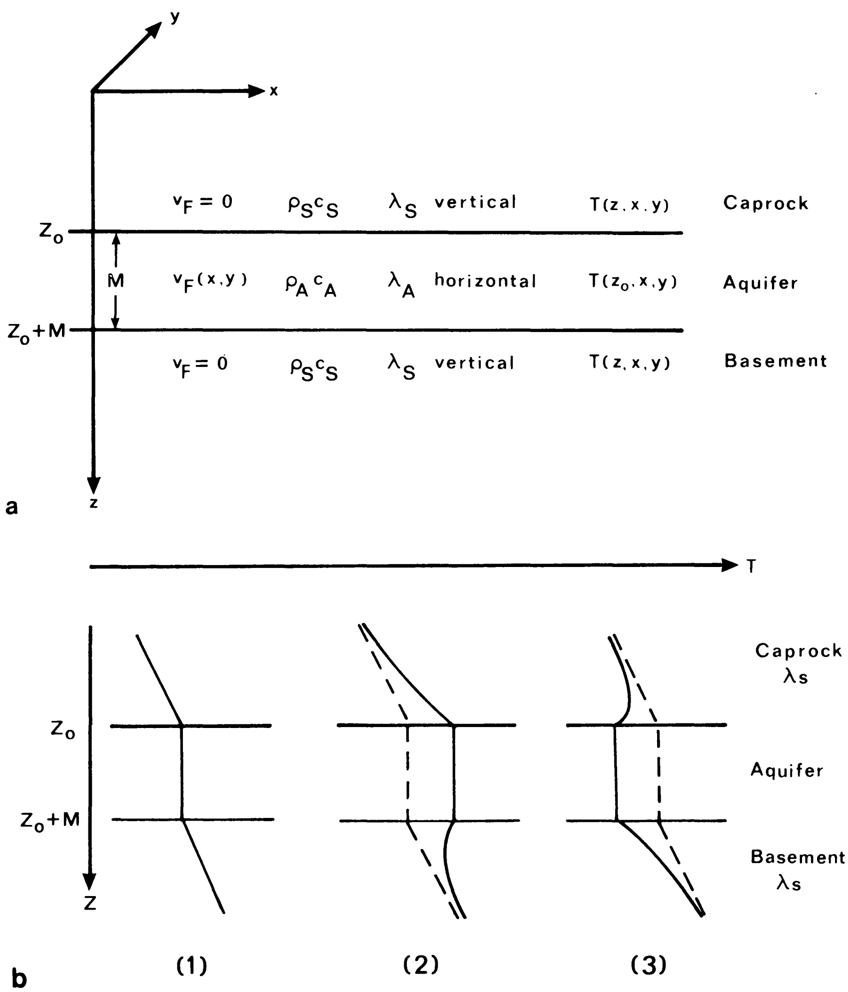

Fig. 1a, b. Hydrogeothermal model.
a Geometry and parameters.
b Vertical energy balance in the aquifer: (1) undisturbed temperature; energy is balanced; (2) injection of warm water; energy is extracted from the aquifer into the caprock and the basement; (3) injection of cold water; energy is extracted from the caprock and the basement into the aquifer

A2) The flow is steady-state.

A3) The spatial derivation of the fluid density vanishes. This is guaranteed if the density is constant or depends on the pressure p only, since in this case

$$\operatorname{grad} \rho_F = \rho_F \beta_F \operatorname{grad} p \approx 0$$

is valid because of the low isotherm compressibility of water ($\beta_F = 5 \times 10^{-10}$ m s^2 kg^{-1}).

Then Eq. (2a) can be reduced to the continuity equation

$$\operatorname{div} \mathbf{q} + \tilde{Q} = 0. \tag{2b}$$

If the heat production can be neglected, the heat transport equation is given by

$$\frac{\partial}{\partial t}(\rho c T) = \operatorname{div}(\lambda \operatorname{grad} T - \rho c T \mathbf{v}) \tag{3a}$$

where

c	(J kg^{-1} K^{-1})	specific heat capacity,
T	(°C)	temperature,
λ	(W m^{-1} K^{-1})	thermal conductivity.

Thermal dispersion and dissipation can be left out of Eq. (3a), as Bear (1972, § 10.7.4/5) describes.

Equation (3a) is reduced to the heat conduction equation assuming an impermeable solid

$$\frac{\partial}{\partial t}(\rho_s c_s T) = \operatorname{div}(\lambda_s \operatorname{grad} T). \tag{3b}$$

In an aquifer two different portions of media have to be considered; therefore, Eq. (3a) can be rewritten as

$$\frac{\partial}{\partial t}(\rho_A c_A T) = \operatorname{div}(\lambda_A \operatorname{grad} T - \rho_F c_F T \mathbf{v}_F) \tag{3c}$$

where

$$\rho_A c_A = n \rho_F c_F + (1-n) \rho_m c_m.$$

(The index m marks the aquifer matrix. $\rho_m \cdot c_m = \rho_s \cdot c_s$ is assumed for the model calculations.)

Model assumptions

In order to calculate the temperature distribution, the heat transfer equation, Eq. (3), has to be solved. The unknown velocity $\mathbf{v}$ must be determined by Darcy's law, Eq. (1), in conjunction with the continuity equation, Eq. (2). In order to solve this problem for a confined aquifer by analytical methods, additional model assumptions are necessary (see Fig. 1a):

A4) All parameters of the model are isotropic and constant. The parameters of the caprock and the basement are identical.

A5) The aquifer is infinite in the horizontal direction; its vertical thickness M is constant.

A6) The caprock and the basement are impermeable, i.e. $\mathbf{v}=0$. In the aquifer, the water flows only in the horizontal direction, i.e. $\mathbf{v}=(v_x, v_y, 0)$.

A7) The temperature in the aquifer is independent of depth.

$T(x, y, z)=T(x, y, z_0)$ for $z_0 \le z \le z_0+M$.

z_0 is the depth of the aquifer top.

A8) The horizontal heat conduction in the surrounding rock is ignored, i.e. the vertical heat conduction is, as usual, dominant.

Taking these assumptions into account, the temperature T in the rock is governed by the heat conduction equation, see Eq. (3b),

$$\rho_s c_s \frac{\partial T}{\partial t}=\lambda_s \frac{\partial^2}{\partial z^2} T \tag{4}$$

and in the aquifer by the modified heat transfer equation, see Eq. (3c),

$$\rho_A c_A \frac{\partial T}{\partial t}+\rho_F c_F \mathbf{v}_F \operatorname{grad} T=\lambda_s \frac{2}{M} \left.\frac{\partial T}{\partial z}\right|_{z=z_0}+\lambda_A \Delta T \tag{5}$$

with the two-dimensional Laplace operator

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

Because of assumption A7, this equation is considered only at the boundary of the aquifer. Since the influence of the whole aquifer has to be taken into account, the conductive part of the energy balance in the z direction need not be considered in a differential sense (see Fig. 1b). The term

$$\frac{\partial}{\partial z}\left(\lambda_A \frac{\partial}{\partial z} T\right)$$

must be replaced by (Myers, 1971; Landel and Sauty, 1978)

$$\lambda_s \frac{2}{M} \left.\frac{\partial T}{\partial z}\right|_{z=z_0}.$$

The unknown velocity $\mathbf{v}_F$ in Eq. (5) is determined by Darcy's law, Eq. (1b); the hydraulic potential ϕ is governed by the two-dimensional potential equation

$$\Delta \phi=\sum_j \tilde{Q}_j M/T_r \tag{6}$$

where

$\tilde{Q}_j$ ($m^3\,s^{-1}\,m^{-3}$) source term for each well,
T_r ($m^2\,s^{-1}$) transmissivity

$$T_r=\int_{z_0}^{z_0+M} k_f\,dz=k_f M.$$

Equation (6) is a combination of Eqs. (1b) and (2b). This approach is based on the assumption that the source term vanishes except at the location of injection or extraction wells.

It is further assumed that the initial temperature T_0 of the aquifer at time $t=0$ is constant. At that time water of temperature T_i is pumped into the aquifer at the injection wells. The flow rate and the temperature of the injected water need not be constant in time.

One-dimensional models

In a one-dimensional model the flow of the groundwater can be considered only in one cartesian direction or in a radial direction. Flow in one horizontal direction can be caused by a number of injection wells lying in a straight line. The temperature field is characterized in the rock by Eq. (4) and in the aquifer by, see Eq. (5),

$$\rho_A c_A \frac{\partial T}{\partial t}+\rho_F c_F v_F \frac{\partial T}{\partial x}=\lambda_s \frac{2}{M} \left.\frac{\partial T}{\partial z}\right|_{z=z_0}+\lambda_A \frac{\partial^2 T}{\partial x^2}. \tag{7}$$

The velocity of the water is given by

$v_F=v_0+Q/(aM)$

where

v_0	($m\,s^{-1}$)	natural Darcy velocity,
Q	($m^3\,s^{-1}$)	injection rate of each well,
a	(m)	distance between two wells.

If the conductive heat transfer is considered ($\lambda_A>0$), the relative temperature differences are given by Avdonin (1964)

$$\frac{T(x, z, t)-T_0}{T_i-T_0} =\frac{4x}{\pi M\sqrt{\tau}} \int_0^1 \frac{\exp\{-[\gamma s\sqrt{\tau}-x/(Ms\sqrt{\tau})]^2\}}{s^2} \cdot \int_{w(s^2)}^{\infty} \exp(-u^2)\operatorname{erfc} \frac{\bar{z}\cdot\lambda_A \varepsilon}{2\lambda_s\sqrt{\tau}\sqrt{1-\frac{\varepsilon^2 s^4 \tau}{4u^2}-s^2}}\,du\,ds \tag{8}$$

where

$\tau =(4\lambda_A t)/(\rho_A c_A M^2)$,
$\gamma =(v_F M c_F \rho_F)/(4\lambda_A)$,
$\varepsilon =\sqrt{(\lambda_s c_s \rho_s)/(\lambda_A c_A \rho_A)}$,
$w(s)=(\varepsilon s\cdot\sqrt{\tau})/(2\sqrt{1-s})$,
$\bar{z} =\begin{cases}0 & \text{for } z_0\le z\le z_0+M\\ 2|z-z_0-M/2|/M-1 & \text{otherwise,}\end{cases}$

T_0	initial temperature at $t=0$,
T_i	temperature of injected water,
s, u	dummy integration parameters,
erfc	complementary error function.

Taking $\bar{z}=0$, the relative temperature differences within the aquifer are given by

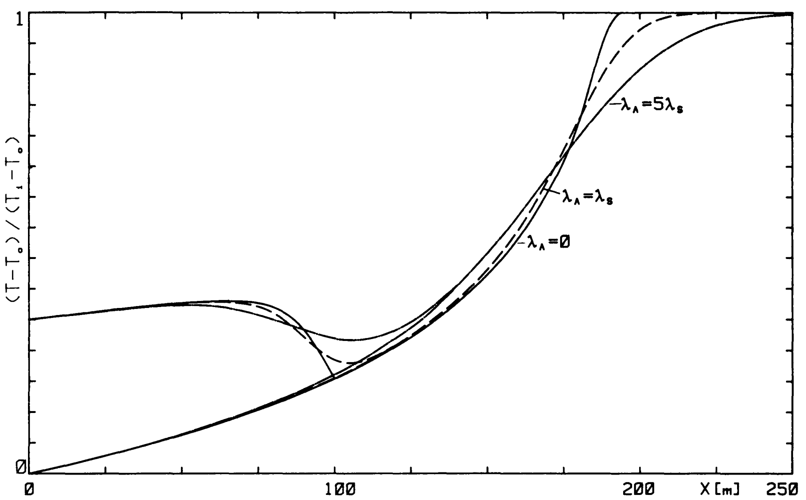

Fig. 2. One-dimensional temperature distribution for a line source after $t=2$ years. The *upper curves* represent an increase of the injection temperature from T_i to $(T_0+2T_i)/3$ after 1 year. $v=1.7\times 10^{-6}$ m s^{-1}, λ_A horizontal thermal conductivity of the aquifer. Other parameters: see Table 1(b)

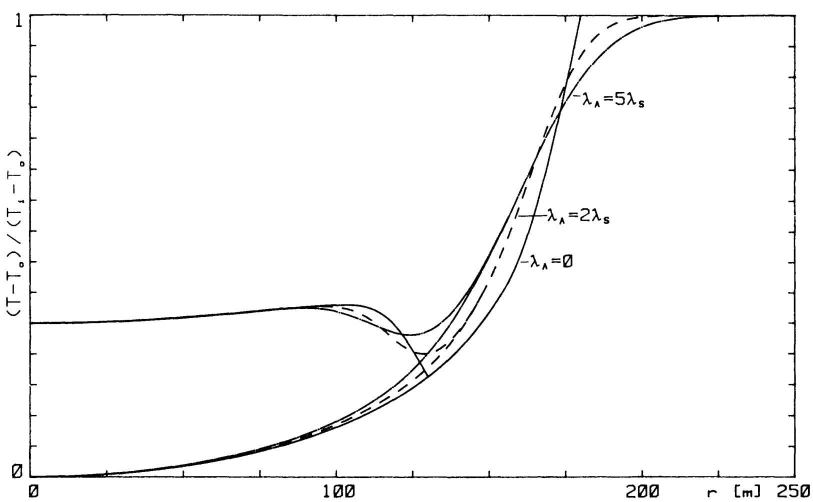

Fig. 3. One-dimensional temperature distribution for a point source (one injection well) after $t=2$ years. The *upper curves* represent an increase of the injection temperature from T_i to $(T_0+2T_i)/3$ after 1 year. λ_A horizontal thermal conductivity of the aquifer. Other parameters: see Table 1(b)

$$\frac{T(x,z_0,t)-T_0}{T_i-T_0} = \frac{2x}{M\sqrt{\pi\tau}}\int_0^1 \frac{\exp\{-[\gamma s\sqrt{\tau}-x/(Ms\sqrt{\tau})]^2)}{s^2} \cdot \operatorname{erfc}\left(\frac{\varepsilon s^2\sqrt{\tau}}{2\sqrt{1-s^2}}\right) ds. \quad (9)$$

If the conductive heat transfer can be neglected ($\lambda_A=0$), the temperature is described by Lauwerier (1955)

$$\frac{T(x,z,t)-T_0}{T_i-T_0} = U(\tau-\xi)\operatorname{erfc}\left[\frac{\xi+\bar{z}}{2\sqrt{\theta(\tau-\xi)}}\right] \quad (10)$$

where

$\tau=(4\lambda_s t)/(\rho_A c_A M^2)$,
$\xi=(4\lambda_s x)/(\rho_F c_F M^2 v_F)$,
$\theta=(\rho_A c_A)/(\rho_s c_s)$,
$\bar{z}$ = as in Eq. (8),
U = unit step function.

Both cited papers give full details about the mathematical derivations of Eqs. (8)–(10). The solution without conductive heat transfer [$\lambda_A=0$, Eq. (10)] cannot be treated as a special case ($\lambda_A\to 0$) of Eq. (8); both solutions need their own approach.

The temperature, depending on the distance to the line source, is plotted for three different values of the thermal conductivity of the aquifer in Fig. 2. The parameters used are given in Table 1. Figure 2 shows the well-known fact that conduction can be neglected if the velocity of the groundwater is high enough; that means, in this case, that the "Peclet number" γ in Eq. (8) should be greater than 4. If a horizontal thermal conductivity in the aquifer is assumed, deviations are observable only at the temperature front.

The solutions given above are less important for practical applications as line sources seldom exist. Therefore, the flow of groundwater in a radial direction, caused by a point source (one well), is analysed. Again the temperature field in the rock is characterized by Eq. (4) and in the aquifer, analogously to Eq. (7), by

$$\rho_A c_A \frac{\partial T}{\partial t} - \frac{\lambda_A}{r}\left(1-\frac{Q c_F \rho_F}{2\pi M \lambda_A}\right)\frac{\partial T}{\partial r} = \lambda_s \frac{2}{M}\left.\frac{\partial T}{\partial z}\right|_{z=z_0} + \lambda_A \frac{\partial^2 T}{\partial r^2} \quad (11)$$

Table 1. Parameters used in model calculations

Parameters		Dimension	(a)	(b)
Specific heat capacity of the rock	c_s	$\mathrm{J\,kg^{-1}\,K^{-1}}$	1000	850
Density of the rock	ρ_s	$\mathrm{kg\,m^{-3}}$	2600	2670
Thermal conductivity of the rock	λ_s	$\mathrm{W\,m^{-1}\,K^{-1}}$	2.8	2.9
Depth of the aquifer	z_0	m	≥ 100	586
Thickness of the aquifer	M	m	30	35
Porosity	n	%	1.0	2.7
Distance of the wells	$2a$	m	400	400
Undisturbed temperature	T_0	°C	40	40
Injection temperature	T_i	°C	25	25
Injected volume flow rate	Q	$\mathrm{m^3\,s^{-1}}$	0.03	0.03

(a) Principal model (Fig. 5)
(b) Parameters for the area of Saulgau (Baden-Württemberg, southern Germany)

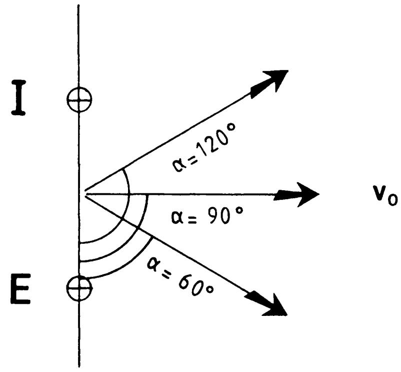

Fig. 4. Principle of a doublet: I injection well at $(-a, 0)$; E extraction well at $(a, 0)$; v_0 natural flow field; α azimuth of the natural flow field

where r is the distance to the well. The solution of Eq. (11) is given by Avdonin (1964)

$$\frac{T(r,z,t)-T_0}{T_i-T_0}=\frac{2}{\sqrt{\pi}\,\Gamma(v)}\left(\frac{r^2}{M^2\tau}\right)^v \int_0^1 \frac{\exp[-r^2/(M^2\tau s)]}{s^{v+1}} \cdot \int_{w(s)}^{\infty} \exp(-u^2)\,\mathrm{erfc}\,\frac{\bar{z}\cdot\lambda_A\varepsilon}{2\lambda_s\sqrt{\tau}\sqrt{1-\frac{\varepsilon^4\tau s^2}{4u^2}-\varepsilon^2 s}}\,du\,ds$$

where

$v=(Q c_F \rho_F)/(4\pi M \lambda_A)$,
$\bar{z}$, ε, τ and w as in Eq. (8),
Γ Gamma function.

If the thermal conductivity is neglected ($\lambda_A=0$), the solution of Eq. (11) is a special case (one well, $v_0=0$) of Eq. (19) below. Both solutions are compared in Fig. 3. It is shown, as in Fig. 2, that heat conduction is negligible.

Two-dimensional model

If the effect of two or more wells and of an additional natural flow of groundwater is taken into account, a two-dimensional approach of the heat transfer within the aquifer is necessary. The flow of the groundwater is described by Darcy's law, Eq. (1), where the hydraulic potential ϕ has to satisfy the potential equation, Eq. (6). This hydraulic problem can by solved by the theory of conformal mapping and by superposition of the effects of each well (Dacosta and Bennett, 1960; Bear, 1972).

The case of a doublet will be considered in particular. A doublet consists of two wells (Fig. 4); in the well I water is injected into the aquifer, in the other well E water is extracted; the filter length is identical to the thickness M of the aquifer. The number of wells has been limited to two only for practical reasons; a generalization to an arbitrary number of wells is possible. If the same injection and extraction rate is assumed ($Q_1=-Q_2=Q$), the solution of the hydraulic problem is given by the velocity potential

$$\varphi(x,y)=-\frac{v_0}{k_f}(x\cos\alpha+y\sin\alpha)-\frac{Q}{4\pi T_r}\ln\frac{(x+a)^2+y^2}{(x-a)^2+y^2}, \quad (13)$$

the stream function

$$\psi(x,y)=-\frac{v_0}{k_f}(y\cos\alpha-x\sin\alpha)-\frac{Q}{2\pi T_r}\arctan\frac{2ay}{a^2-x^2-y^2} \quad (14)$$

and the velocity

$$v_x=v_0\cos\alpha+\frac{Q}{2\pi M}\left[\frac{x+a}{(x+a)^2+y^2}-\frac{x-a}{(x-a)^2+y^2}\right]$$
$$v_y=v_0\sin\alpha+\frac{Q}{2\pi M}\left[\frac{y}{(x+a)^2+y^2}-\frac{y}{(x-a)^2+y^2}\right] \quad (15)$$

where

$2a$ the distance between the two wells,
v_0 velocity of the natural flow field,
α azimuth of the natural flow field (see Fig. 4).

The temperature field in the rock is characterized by the heat conduction equation, Eq. (4), and in the aquifer by the heat transfer equation, see Eq. (5),

$$\rho_A c_A \frac{\partial T}{\partial t}+\rho_F c_F \mathbf{v}\,\mathrm{grad}\,T=\lambda_s\frac{2}{M}\frac{\partial T}{\partial z}\bigg|_{z=z_0}. \quad (16)$$

As shown above, the conductive heat transfer can be neglected for sufficiently great injection rates.

If the velocity $\mathbf{v}$, Eq. (15), is described as a function of φ and ψ, it is well-known that $\mathbf{v}$ depends on the potential φ but does not depend on the stream function ψ. Therefore, the dependency on the two spatial coordinates x and y can be reduced to one coordinate. The following equation is valid in the new $\varphi-\psi$ coordinate system:

$$\mathbf{v}\,\mathrm{grad}\,T=-\frac{v^2}{k_f}\frac{\partial T}{\partial\varphi}. \quad (17)$$

Opposed to this approach, Gringarten and Sauty (1975) have used the area element S as parameter

$$\mathbf{v}\,\mathrm{grad}\,T=k_f\,\Delta\psi\,\frac{\partial T}{\partial S}. \quad (18)$$

This approach has often been cited (e.g. Mercer et al., 1982), but has the disadvantage of using a relatively long comput-

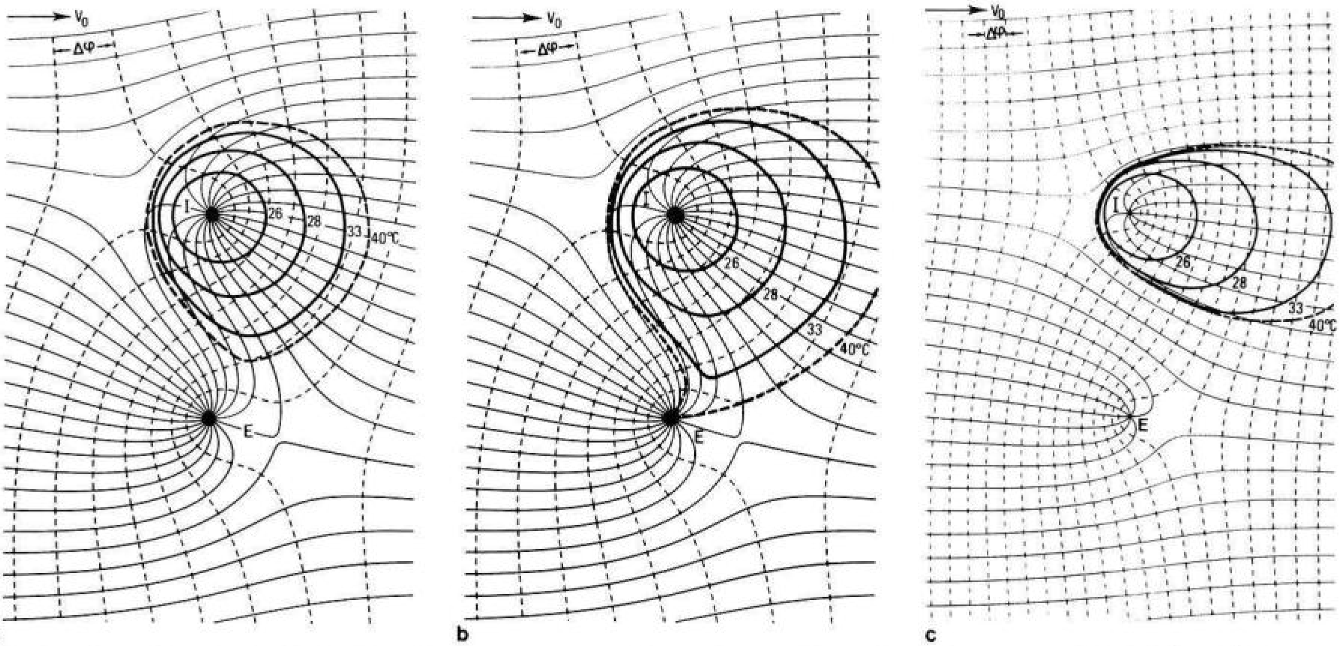

Fig. 5a–c. Two-dimensional distribution of temperature, streamlines and potential. **a** after $t=3$ years, $v_0=1.0\times10^{-6}$ m s^{-1}; **b** after $t=5$ years, $v_0=1.0\times10^{-6}$ m s^{-1}; **c** after $t=3$ years, $v_0=2.5\times10^{-6}$ m s^{-1}. $\alpha=90°$ azimuth. Other model parameters: see Table 1(a)

ing time. The new approach, Eq. (17), seems to be clearer from the mathematical point of view and the computing time can be reduced to one-tenth. The derivation of the solution with the new approach is given in the Appendix. The temperature distribution at time t and point $\varphi_0=\varphi(x_0, y_0)$ is given by

$$\frac{T(t,\varphi_0,z)-T_0}{T_i-T_0}=U\left[t-\frac{\rho_A c_A}{\rho_F c_F}I(\varphi_0)\right] \cdot \operatorname{erfc}\frac{\frac{\lambda_s}{\mathrm{M}\rho_F c_F}\mathrm{I}(\varphi_0)+\frac{\mathrm{M}}{4}\bar{z}}{\sqrt{\frac{\lambda_s}{\rho_s c_s}}\sqrt{t-\frac{\rho_A c_A}{\rho_F c_F}I(\varphi_0)}} \tag{19}$$

where

$$I(\varphi_0)=k_f\int_{\varphi_0}^{\infty}\frac{1}{v^2}\,d\varphi, \tag{20}$$

$\bar{z}$ as in Eq. (8).

The integration contour in Eq. (20) is the streamline $\psi_0=\psi(x_0, y_0)$. If we assume that the natural groundwater flow vanishes ($v_0=0$), the integral, Eq. (20), can be given in a completely analytical way. Following the approach of Muskat (cited by Bear, 1979) the two parameters ζ and ϑ are defined as

$$\zeta=\tfrac{1}{2}\ln\frac{(x-a)^2+y^2}{(x+a)^2+y^2} \tag{21}$$

$$\vartheta=\arctan\frac{2ay}{a^2-x^2-y^2}. \tag{22}$$

The integral I, Eq. (20), can be rewritten as

$$I(\zeta_0)=\frac{2\pi M a^2}{Q}\int_{\zeta_0}^{\infty}(\cosh\zeta+\cos\vartheta)^{-2}\,d\zeta. \tag{23}$$

Using integration formulas, e.g. Gradshteyn and Ryzhik (1965), the following result is yielded

a) $|\cos\vartheta|\neq1$

$$I(\zeta)=\frac{2\pi M a^2}{Q}\left\{1+\left[2\cot\vartheta\cdot\arctan\left(\tanh\frac{\zeta}{2}\tan\frac{\vartheta}{2}\right)-\frac{\sinh\zeta}{\cosh\zeta+\cos\vartheta}\right]\sin^{-2}\vartheta\right\} \tag{24a}$$

b) $|\cos\vartheta|=1$, i.e. $y=0$

$$I(\zeta)=\frac{2\pi M a^2}{3Q}\left\{1-\frac{\sinh\zeta}{\cosh\zeta+\cos\vartheta}\left(1+\frac{\cos\vartheta}{\cosh\zeta+\cos\vartheta}\right)\right\}. \tag{24b}$$

If the natural flow of groundwater does not vanish ($v_0\neq0$), the approach of Muskat cannot be used. The integral, Eq. (20), has to be treated numerically. For a given point (x_0, y_0) the stream function $\psi_0=\psi(x_0, y_0)$, Eq. (14), the potential $\varphi_0=\varphi(x_0, y_0)$, Eq. (13), and the velocity $\mathbf{v}(x_0, y_0)$, Eq. (15), have to be calculated. Then the potential φ_0 is increased by a small value $\Delta\varphi$ to $\varphi_0+\Delta\varphi$; this is a new point on the integration contour. The corresponding cartesian $x-y$ coordinates are given by an implicit system of equations

$$\begin{aligned}\psi_0&=\psi(x,y)\\ \varphi_0+\Delta\varphi&=\varphi(x,y).\end{aligned} \tag{25}$$

Since ψ and φ are nonlinear function, a direct inversion of the system (25) is not possible. In order to solve the system, the Newton-Raphson method can be used (e.g. Stoer, 1972):

$$\begin{bmatrix} x_{(m+1)} \\ y_{(m+1)} \end{bmatrix} = \begin{bmatrix} x_{(m)} \\ y_{(m)} \end{bmatrix} - \begin{bmatrix} \partial\psi/\partial x & \partial\psi/\partial y \\ \partial\varphi/\partial x & \partial\varphi/\partial y \end{bmatrix}^{-1} \begin{bmatrix} \psi(x_{(m)}, y_{(m)}) - \psi_0 \\ \varphi(x_{(m)}, y_{(m)}) - \varphi_0 - \Delta\varphi \end{bmatrix}.$$

Using the relationship of φ, ψ and v, Eqs. (13)–(15), the two-dimensional system can be rewritten

$$\begin{bmatrix} x_{(m+1)} \\ y_{(m+1)} \end{bmatrix} = \begin{bmatrix} x_{(m)} \\ y_{(m)} \end{bmatrix} + \frac{k_f}{v^2} \begin{bmatrix} -v_y & v_x \\ v_x & v_y \end{bmatrix} \begin{bmatrix} \psi(x_{(m)}, y_{(m)}) - \psi_0 \\ \varphi(x_{(m)}, y_{(m)}) - \varphi_0 - \Delta\varphi \end{bmatrix}. \quad (26)$$

Since the convergence of the Newton-Raphson method is very fast, only a small number of iterations is needed to determine the x_1 and y_1 values belonging to $\varphi_0 + \Delta\varphi$; the corresponding value $v^2(x_1, y_1)$ is calculated from Eq. (15). The procedure is continued for $\varphi_0 + 2\Delta\varphi$, starting the new iteration, Eq. (26), with the old $x_1 - y_1$ values.

For sufficiently great N, $\varphi = \varphi_0 + N\Delta\varphi$, the point (x_N, y_N) should lie in the neighbourhood of the injection well: $(x_N, y_N) \approx (-a, 0)$. Then the integral, Eq. (20), is numerically calculated by

$$I(\varphi_0) = k_f \sum_{i=0}^{N} \frac{\Delta\varphi}{v^2(x_i, y_i)}.$$

Otherwise the point (x_0, y_0) is situated at a streamline ψ_0 which does not lead to the injection well; in this case we set

$$I(\varphi_0) = \infty$$

and therefore, see Eq. (19),

$$T(t, \varphi_0, z) = T_0 \quad \text{for all } t.$$

The Newton-Raphson method is also used for the representation of stream and potential lines. These lines, together with the isotherms, are plotted in Fig. 5; the parameters used are given in Table 1(a). In Fig. 5a and b the velocity of the natural groundwater flow is $v_0 = 1 \times 10^{-6}$ m s^{-1} and its azimuth (see Fig. 4) is 90°. In Fig. 5a the temperature distribution is shown after a working time for the doublet of 3 years. The course of the temperature front is given by the isotherm of 40° C. This front will arrive at the extraction well after less than 5 years, see Fig. 5b. If the velocity is slightly raised to 2.5×10^{-6} m s^{-1}, there is no single streamline connecting both wells, as shown in Fig. 5c. As a steady-state flow is assumed, a thermal connection between both wells can be excluded for this model; see also Fig. 6.

Temperature in the extraction well

In geothermal energy applications, the behaviour of the temperature in the extraction well at $(a, 0)$ is decisive. One problem is whether, and when, the temperature front of the injected water will arrive at the extraction well. As we have seen in Eq. (19) the argument of the unit step function is important. If this argument is greater than zero, the temperature front has arrived at the extraction well. This data is called the thermal breakthrough time:

$$t_B = \frac{\rho_A c_A}{\rho_F c_F} I(\varphi_a) \quad (27)$$

Table 2. Thermal breakthrough times for a doublet

Natural flow of groundwater		Thermal breakthrough time
Velocity v_0	Azimuth α	t_B
$v_0 = 0$	–	$\frac{\rho_A c_A}{\rho_F c_F} \frac{4\pi M a^2}{3Q}$
$v_0 > 0$	0°	$\frac{\rho_A c_A}{\rho_F c_F} \frac{2a}{v_0} \left(1 - \frac{Q}{2\pi M v_0 c} \ln \frac{c+a}{c-a}\right)$
$Q/(a\pi M) > v_0 > 0$	180°	$\frac{\rho_A c_A}{\rho_F c_F} \frac{2a}{v_0} \left(-1 + \frac{Q}{\pi M v_0 c} \arctan \frac{a}{c}\right)$
$v_0 \geq Q/(a\pi M)$	180°	∞

Q (m^3 s^{-1}) injection and extraction rate; $2a$ (m) distance between both wells; M(m) thickness of the aquifer; $c = |a^2 + aQ/(\pi M v_0 \cos\alpha)|^{1/2}$

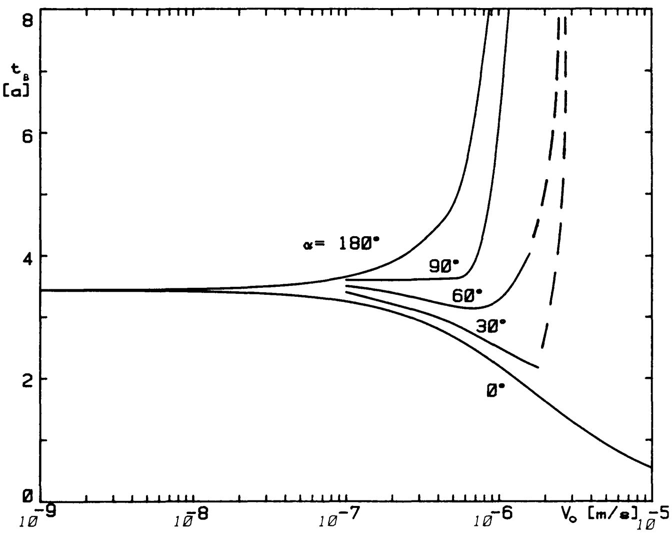

Fig. 6. Thermal breakthrough time t_B for a doublet: v_0 natural flow field; α azimuth. Other model parameters: see Table 1(b)

where $I(\varphi_a)$ is the lowest value of the integral, Eq. (20), relating to all streamlines ψ connecting both wells. If there is no natural flow field ($v_0 = 0$), the straight line IE (see Fig. 4) is the streamline that has been searched for. In this case the integral can be calculated by using Eq. (24b) with $\cos\vartheta = -1$ and Eq. (21)

$$I(x_0) = \frac{\pi M}{3Qa}(2a^3 + 3a^2 x_0 - x_0^3)$$

where the point $(x_0, 0)$ is situated on IE ($|x_0| \leq a$). This result is well known (e.g. Lippmann and Tsang, 1980). The straight line IE is also the streamline we have been looking for, if the direction of the velocity is parallel to IE ($\alpha = 0°$ or $\alpha = 180°$). Using Eqs. (13)–(15) the integral is given by ($|x_0| \leq a$)

$$I(x_0) = \pi M \int_{-a}^{x_0} (a^2 - x^2)/[aQ + \pi M v_0 (a^2 - x^2) \cos\alpha]\, dx.$$

Setting $x_0 = a$, the thermal breakthrough time can be calculated by simple formulas which are summarized in Table 2.

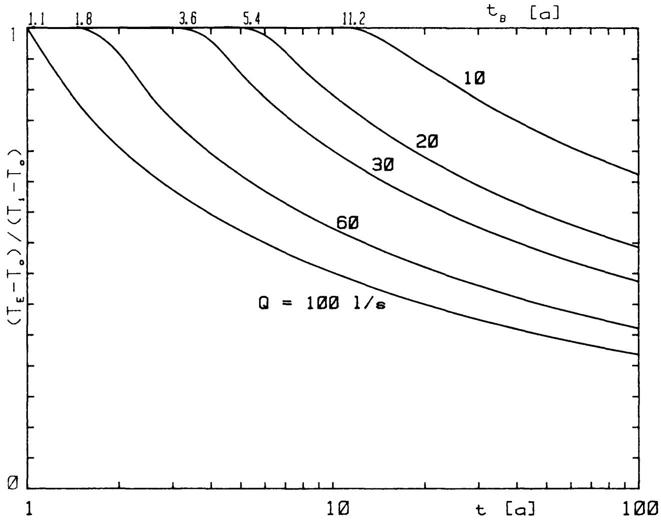

Fig. 7. Extraction temperature T_E for a doublet: t working time of the doublet; t_B thermal breakthrough time; Q injection-extraction rate; $v_0 = 1.0 \times 10^{-7}$ m s^{-1} (natural flow field); $\alpha = 90°$ (azimuth). Other model parameters: see Table 1(b)

The breakthrough times for all other cases can be numerically determined as described above. The breakthrough times, depending on the velocity v_0 and the azimuth, are plotted in Fig. 6. They are identical for vanishing and low velocities. For higher velocities, higher than 10^{-7} m s^{-1} for the given parameters, the azimuth is important.

The breakthrough time is the date of the first connection of the injected water with the extraction well. After this time there is not necessarily an important change in the extraction temperature T_E. This temperature is a mixed temperature integrated over all streamlines connected with the extraction well. It is given by, see Eq. (19),

$$\frac{T_E(t) - T_0}{T_i - T_0} = \frac{1}{2\pi} \int_{\psi_a}^{\psi_a + 2\pi} U\left(t - \frac{\rho_A c_A}{\rho_F c_F} I_\psi\right) \cdot \operatorname{erfc}\left(\frac{\sqrt{\lambda_s \rho_s c_s}}{M \rho_F c_F} I_\psi \Big/ \sqrt{t - \frac{\rho_A c_A}{\rho_F c_F} I_\psi}\right) d\psi \tag{28}$$

with

$$\psi_a = \frac{v_0}{k_f} a \sin\alpha$$

$$I_\psi = k_f \int_{-\infty}^{+\infty} \frac{1}{v^2} d\varphi_\psi$$

where the integration contour is the streamline ψ.

The extraction temperature for different injection-extraction rates, depending on the working time of a doublet, is plotted in Fig. 7. For example, the breakthrough time for an extraction rate of 0.03 m^3 s^{-1} (30 l s^{-1}) in the given case is about 3 years and 2.5 months, but a marked reduction of the temperature by 33% will occur after about 15 years. This shows that the economical working time of a doublet can be considerably longer than the thermal breakthrough time.

Remarks

The formulas given in this paper can also be used for the temperature distribution in the caprock and in the basement. But in this case the aquifer must be situated at greater depth, since the solutions are given for a full space without boundary conditions at the earth's surface. For this case Kasamyer et al. (1984) give a solution for the one-dimensional model. Chen (1980) shows that the temperature distribution in the aquifer is independent of the caprock thickness z_0 for a time period t after injection starts, if

$$t \le 0.1\, z_0^2\, \rho_s c_s / \lambda_s.$$

This relation is always valid for aquifers deeper than 100 m and a working time of a doublet of 30 years.

Appendix

Solution of the differential equations

We have to find the solution of the heat conduction equation, Eq. (4), in the basement

$$\rho_s c_s \frac{\partial T}{\partial t} = \lambda_s \frac{\partial^2 T}{\partial z^2}; \qquad z > z_0 + M. \tag{29}$$

The temperature T has to satisfy the boundary condition, Eqs. (16) and (17), at $z = z_0 + M$

$$\rho_A c_A \frac{\partial T}{\partial t} - \rho_F c_F \frac{v^2}{k_f} \frac{\partial T}{\partial \varphi} = \lambda_s \frac{2}{M} \frac{\partial T}{\partial z}\bigg|_{z = z_0 + M}. \tag{30}$$

An analogous formulation is valid for the caprock ($z < z_0$). The initial condition is

$$\begin{aligned} T &= 0 \quad \text{for } t = 0 \\ T &= T_i \quad \text{at } (x, y) = (-a, 0);\ \text{i.e. } \varphi = \infty \end{aligned} \tag{31}$$

As the problem is linear, the effects of other wells and of a change of the injection temperature can be considered by superposition.

The differential equations, Eqs. (29) and (30), are transformed by

$$\begin{aligned} t &= (M^2 \rho_A c_A \tau)/(4\lambda_s) \\ \varphi &= (M^2 \rho_F c_F \xi)/(4\lambda_s) \\ z &= \frac{M}{2}(\eta + 2) + z_0 \end{aligned} \tag{32}$$

into the equations

$$\frac{\partial T}{\partial \tau} = c_1 \frac{\partial^2 T}{\partial \eta^2} \qquad \eta > 0 \tag{33}$$

$$\frac{\partial T}{\partial \tau} = \frac{\partial T}{\partial \eta} + \frac{v^2}{k_f} \frac{\partial T}{\partial \xi} \qquad \eta = 0 \tag{34}$$

with

$$c_1 = (\rho_A c_A)/(\rho_s c_s).$$

The Laplace transform (e.g. Abramowitz and Stegun, 1964)

$$L(s, \xi, \eta) = \int_0^\infty e^{-s\tau} T(\tau, \xi, \eta)\, d\tau$$

yields an ordinary differential equation instead of the partial differential equation (33)

$$sL = c_1 \frac{\partial^2}{\partial \eta^2} L. \tag{35}$$

The solution of Eq. (35) is

$$L(s, \xi, \eta) = b_1(\xi, s) \exp(-\sqrt{s/c_1}\,\eta) + b_2(\xi, s) \cdot \exp(\sqrt{s/c_1}\,\eta). \tag{36}$$

Since the second term becomes infinite if $\eta \to \infty$, i.e. $z \to \infty$, the coefficient b_2 must vanish. The coefficient b_1 is governed by the equation

$$(s + \sqrt{s/c_1})\, b_1 = \frac{v^2}{k_f} \frac{\partial b_1}{\partial \xi} \tag{37}$$

which is a combination of the Laplace-transformed Eq. (34) and Eq. (36). Its solution is

$$b_1(\xi, s) = b_3(s) \exp[-(s + \sqrt{s/c_1})\, I(\xi)] \tag{38}$$

with

$$I(\xi) = k_f \int_{\xi}^{\infty} \frac{1}{v^2}\, d\bar{\xi}. \tag{39}$$

Using the initial condition, Eq. (31), the coefficient b_3 is determined. The Laplace transform yields for $\eta = 0$ and $\xi = \infty$

$$L(s, \infty, 0) = T_i/s$$

and Eqs. (36) and (38) yield

$$L(s, \infty, 0) = b_3(s).$$

Therefore, the solution of Eq. (35) is given by

$$L(s, \xi, \eta) = T_i \exp\{-I(\xi)\, s + [I(\xi) + \eta] \sqrt{s/c_1}\}/s$$

and the inverse Laplace transform yields the solutions of Eqs. (33) and (34).

$$T(\tau, \xi, \eta) = T_i\, U[\tau - I(\xi)] \cdot \operatorname{erfc} \frac{\eta + I(\xi)}{2\sqrt{c_1}\sqrt{\tau - I(\zeta)}}. \tag{40}$$

Using the inverse transform of Eq. (32), the solution of the original problem is obtained [see Eq. (19)].

Acknowledgements. The author wants to thank Dr. H. Rodemann, Federal Institute of Geoscience and Natural Resources (BGR), for helpful discussions, W. Michel for discussions and programming the solution of Gringarten, and C. Clauser, Technical University of Berlin, for useful hints. The model calculations have been carried out on Cyber 76 at the Regional Computer Centre of Lower Saxony (RRZN), Hannover, and on a personal computer HP 9826. This paper has been supported by the Federal Ministry of Research and Technology (BMFT), Bonn, under project no. 03 E–6240 B.

References

Abramowitz, M., Stegun, I.A.: Handbook of mathematical functions. New York: Dover Publications 1964

Avdonin, N.A.: O nekotorych formulach dlja rasceta temperaturnoyo polja plasta pri teplovoy inzekcii. Neft i gaz **3**, 37–41, 1964 (in Russian)

Bear, J.: Dynamics of fluids in porous media. New York: American Elsevier 1972

Bear, J.: Hydraulics of groundwater. New York: McGraw-Hill 1979

Carslaw, H.S., Jaeger, J.C.: Conduction of heat in solids. Oxford: Oxford University Press 1959

Chen, C.-S.: Temperature distribution around a well during thermal injection and a graphical determination of thermal properties of the aquifer. Ph. D. Thesis, Texas A and M University, 1980

Dacosta, J.A., Bennett, R.R.: The pattern of flow in the vicinity of a recharging and discharging pair of wells in an aquifer having areal parallel flow. Int. Ass. Sci. Hydrol. Publ. **52**, 524–536, 1960

Gradshteyn, I.S., Ryzhik, I.M.: Table of integrals, series, and products. New York: Academic Press 1965

Gringarten, A.C., Sauty, J.P.: A theoretical study of heat extraction from aquifers with uniform regional flow. J. Geophys. Res. **80**, 4956–4962, 1975

Haenel, R.: Efforts for utilizing geothermal law enthalpy water in the Federal Republic of Germany. Deuxiemes Journees Internatinales de Geothermie Appliquee (JIGA 85), 169–180. Bordeaux: L'Agence Francaise pour la Maitrise de l'Energie 1985

Kasamyer, P.W., Younker, L.W., Hanson, J.M.: Development and application of a hydrothermal model for the Salton Sea Geothermal Field, California. Geol. Soc. Am. Bull. **95**, 1242–1252, 1984

Landel, P.-A., Sauty, J.P.: Etude de l'influence des caracteristiques physiques de l'aquifere et des roches encaissantes sur la temperature de l'eau au puits de production d'un doublet hydrothermique. Rapport final, Projet G/C7, Contrat no. 093-76 EGF, Brüssel, 1978

Lauwerier, H.A.: The transport of heat in an oil layer caused by the injection of hot fluid. Appl. Sci. Res., Sect. A, **5**, 145–150, 1955

Lippmann, M.J., Tsang, C.F.: Ground-water use for cooling; associated aquifer temperature changes. Ground Water **18**, 452–458, 1980

Lütkestratkötter, H.: Numerische Behandlung von Wärmeausbreitungsvorgängen in durchströmten porösen Medien nach der Methode der finiten Elemente. Mitteilungen Institut für Wasserbau und Wasserwirtschaft, RWTH Aachen, Heft **20**, 1977

Mehlhorn, H.: Temperaturveränderungen im Grundwasser durch Brauchwassereinleitungen. Mitteilungen Institut für Wasserbau, Universität Stuttgart, Heft **50**, 1982

Mercer, J.W., Faust, C.R., Miller, W.J., Pearson, F.J.: Review of simulation techniques for aquifer thermal energy storage (ATES). Advances in Hydroscience **13**, 2–129, 1982

Myers, G.E.: Analytical methods in conduction heat transfer. New York: McGraw-Hill 1971

Stoer, J.: Einführung in die numerische Mathematik I. Berlin: Springer 1972

Received May 5, 1986 / Revised November 14, 1986
Accepted December 4, 1986